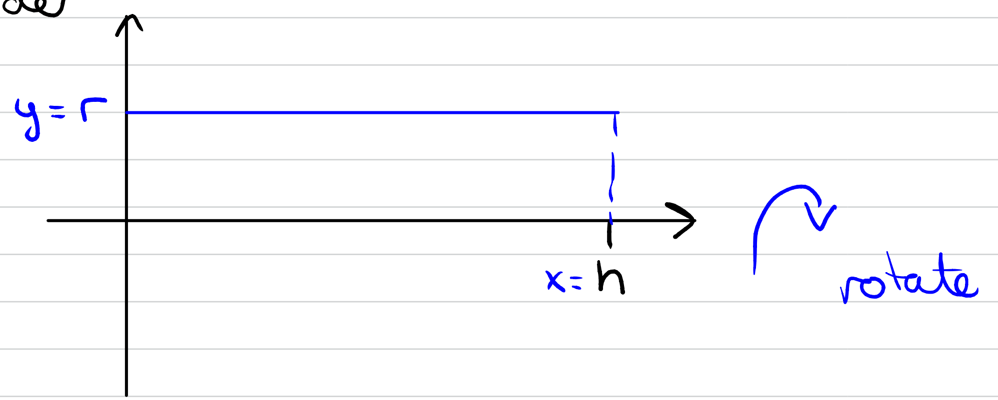
Today's Topic: Volumes

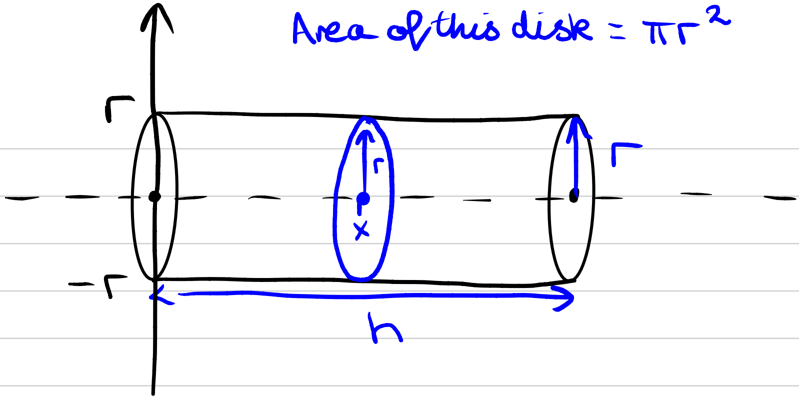
We learnt how to calculate area, but what about volumes?

Consider: cylinder of radius r, length h.

How can we translate this into a question involving integrals?

Consider

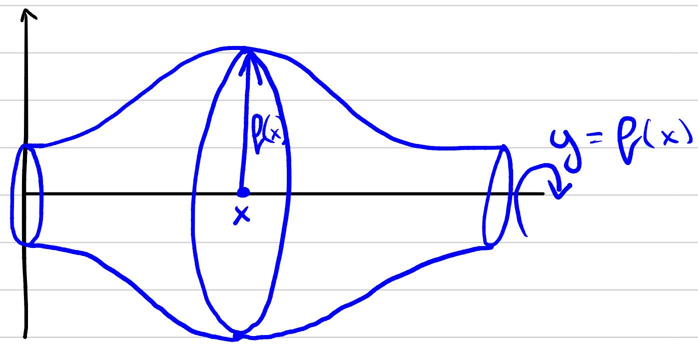




How to calculate volume?

Idea: we take a slice at each x-value, calculate the area, and add them up.

We can also use this idea for more complicated shapes:



This is called the "disk method".

$$V = \int_a^b A(x)\,dx$$

$A(x)$ = cross sectional area at x.

Example: Cylinder.

$A(x) = \pi r^2.$

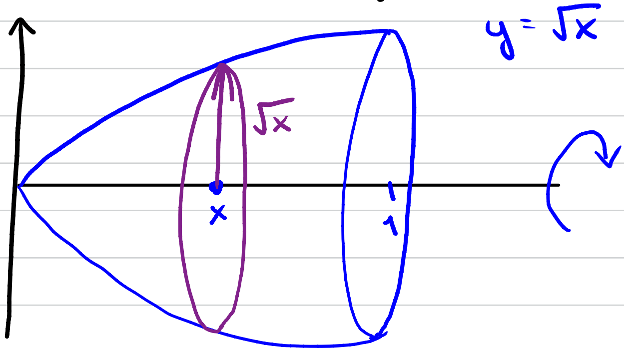
$$V = \int_0^h \pi r^2\, dx = \pi r^2 x \Big|_0^h$$
$$= \pi r^2 h.$$

In general: at each x, we get disk with radius $y = f(x)$.

So $A(x) = \pi (f(x))^2.$

Thus $$V = \int_a^b \pi (f(x))^2\, dx.$$

Example: Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

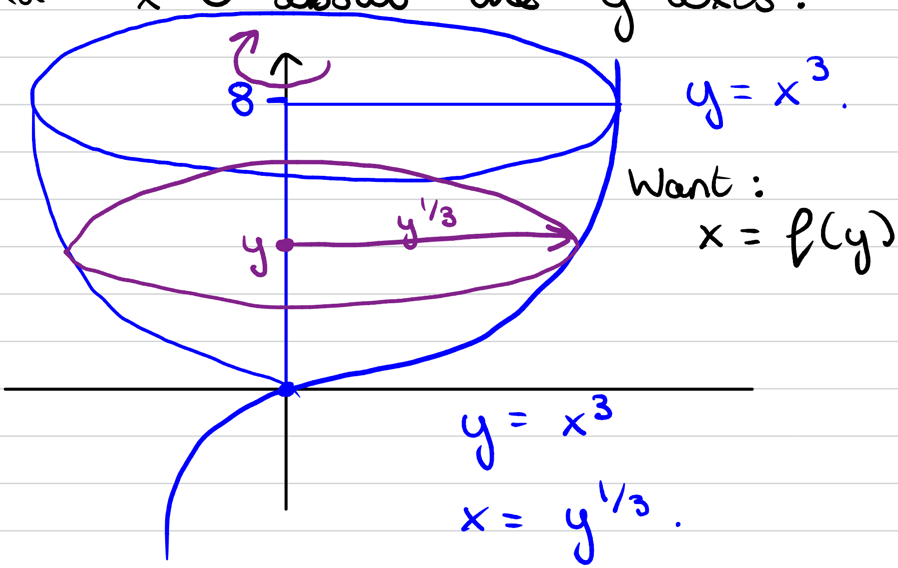


Area of each disk:

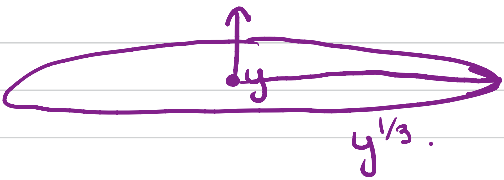
$$A(x) = \pi (f(x))^2 = \pi (\sqrt{x})^2 = \pi x.$$

So $$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x \; dx$$
$$= \pi \left.\frac{x^2}{2}\right|_0^1$$
$$= \frac{\pi}{2}.$$

Example: Find volume of solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.



We want to take horizontal slices.



Cross-sectional area:

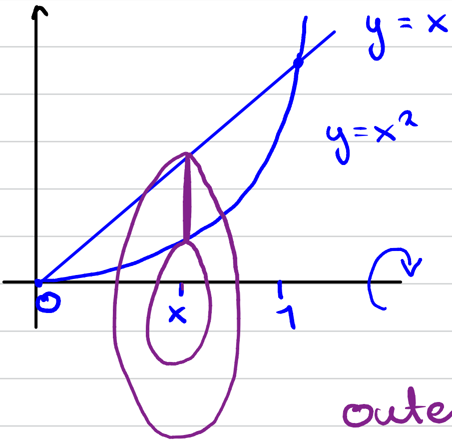
$$A(y) = \pi (y^{1/3})^2 = \pi y^{2/3}.$$

$$V = \int_a^b A(x)\,dx. \qquad V = \int_a^b A(y)\,dy.$$

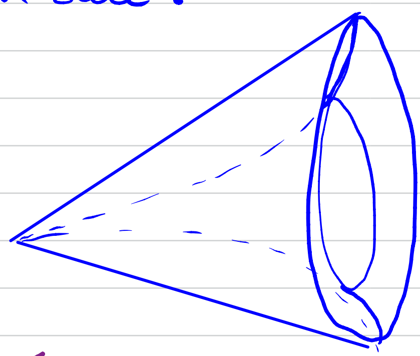
$$V = \int_0^8 \pi y^{2/3}\,dy$$

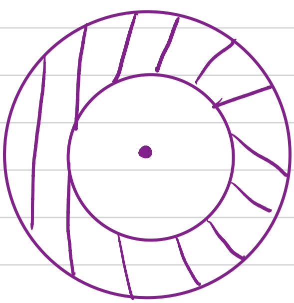
$$= \pi \left[\frac{3y^{5/3}}{5} \right]_0^8 = \frac{96\pi}{5}.$$

Example: Find the volume of the solid obtained by rotating the area enclosed by $y = x$; $y = x^2$ about the x-axis



Each slice:





outer radius $= x$

inner radius $= x^2$.

$A(x) =$ area of outer disk − area of inner disk.

$$= \pi x^2 - \pi (x^2)^2$$
$$= \pi (x^2 - x^4).$$

$$V = \int_a^b A(x)\, dx$$
$$= \int_0^1 \pi (x^2 - x^4)\, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

$$= \frac{2\pi}{15}.$$