Todays Topic: The Substitution Rule

It's important to be able to find antiderivatives in order to evaluate integrals.

Consider
$$\int 2x\sqrt{1+x^2}\,dx$$

* we can't quite integrate
* Goal: to simplify until we can.

Method: a substitution.

Let $u = 1+x^2$.

Taking derivatives: $du = 2x\,dx$

Now: $\int \sqrt{1+x^2}\cdot 2x\,dx = \int \sqrt{u}\,du$

$$= \int u^{1/2}\,du$$

$$= \frac{2u^{3/2}}{3} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

In general, this method works for integrals of the form:
$$\int f(g(x))\,g'(x)\,dx$$

Recall: $\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x))\,g'(x)$

Let $u = g(x)$. Then $du = g'(x)\,dx$,

and $$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Note: this should remind you of the chain rule. - in fact, this is exactly the inverse process.

Substitution Rule: If $u = g(x)$ is a differentiable function, and f is a continuous function, then:

$$\boxed{\int f(g(x))g'(x)\,dx = \int f(u)\,du}$$

Example: Find $\int x^3 \cos(x^4+2)\,dx$.

We can't integrate $\cos(x^4+2)$, but can integrate something like $\cos(u)$.

Let $u = x^4 + 2$
$du = 4x^3\,dx$, so $\frac{1}{4}du = x^3\,dx$.

Now $$\int x^3\cos(x^4+2)\,dx = \int \cos(u) \cdot \frac{1}{4}\,du$$
$$= \frac{1}{4}\int \cos(u)\,du$$
$$= \frac{1}{4}\sin(u) + C$$

$$= \frac{1}{4}\sin(x^4+2) + C.$$

Main challenge: finding the "correct" substitution.

Comes with practice!

Try something: if it makes the integral more complicated, it wasn't the best choice - try another!
This is common.

Example: Find $\int \sqrt{2x+1}\, dx$.

- Let $u = 2x+1$. Then $du = 2\,dx$
  so $dx = \frac{1}{2}\, du$.

- $$\int \sqrt{2x+1}\, dx = \int \sqrt{u}\, \frac{1}{2}\, du$$

$$= \frac{1}{2}\int u^{1/2}\, du$$

$$= \frac{u^{3/2}}{3} + C$$

$$= \frac{(2x+1)^{3/2}}{3} + C.$$

Note: We could have chosen $u = \sqrt{2x+1}$.
Then $du = \frac{dx}{\sqrt{2x+1}}$, and $dx = \sqrt{2x+1}\, du$
$= u\, du$.

Then $\int \sqrt{2x+1}\,dx = \int u \cdot u\,du = \int u^2\,du$

If you continue, you will get the same answer (check for yourself!).

Example: $\int \tan x\,dx$.

Not immediately of the right form:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

- Let $u = \cos x$, then $du = -\sin x\,dx$
- $\int \tan x\,dx = \int -\frac{du}{u}$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C.$$

<u>Evaluating definite Integrals with substitution:</u>

2 methods:

1. evaluate indefinite integral first, then evaluate at limits.

2. change the limits when we do the sub.

Let us illustrate method 2:

Find
$$\int_1^2 \frac{dx}{(3-5x)^2}.$$

- Let $u = 3-5x$. Then $du = -5\,dx$
  $dx = -\frac{1}{5}\,du$
- Limits: when $x=1$, $u = 3-5\cdot 1 = -2$
  when $x=2$, $u = 3-10 = -7$.
- $$\int_1^2 \frac{dx}{(3-5x)^2} = \int_{-2}^{-7} \frac{1}{u^2}\cdot\left(-\frac{1}{5}\right) du.$$
  $$= -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\, du$$
  $$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7}$$
  $$= -\frac{1}{5}\left(\frac{1}{7} - \frac{1}{2}\right) = \frac{1}{14}.$$