Today's Topic: Integration by Parts

- The FTC tells us that differentiation and integration are inverse to each other.
- every differentiation rule must have a integration rule that reverses the process.

chain rule <=> substitution

product rule <=> integration by parts.

Recall: Product rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + f'(x)g(x)$$

Integrate both sides:

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$$

Rearranging:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Let $u = f(x)$, $v = g(x)$.

Then $dv = g'(x)dx$, and $du = f'(x)du$.

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Example: Find $\int x\sin x\,dx$.

- Step 1: choose your u and dv.

Let: $u = x$, $dv = \sin x\,dx$

Then: $du = dx$, $v = \int \sin x\,dx = -\cos x$.

- Step 2: Apply formula

$$\underset{\int u\,dv}{\underset{\parallel}{\int x\sin x\,dx}} = uv - \int v\,du$$

$$= -x\cos x - \int -\cos x\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C.$$

Main difficulty: choosing the right u and dv.

Aim: to make the $\int v\,du$ to be simpler than we started with.

e.g if we let $u = \sin x$, $dv = x\,dx$

$du = \cos x\,dx$, $v = \frac{x^2}{2}$.

$\int x \sin x \, dx$

and

$$uv - \int v \, du = \frac{x^2}{2}\sin x - \int \frac{x^2}{2}\cos x \, dx$$

Worse than before!

- if this happens, goback to beginning and try a different choice.

Example: Find $\int \ln x \, dx . = \int \ln x \cdot 1 \, dx$

Step 1: $u = \ln x \qquad dv = 1 . dx$

$du = \frac{1}{x} dx \qquad v = x$

Step 2: $\int \ln x \, dx = uv - \int v \, du$

$$= (\ln x) x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

Example (more than once): Find $\int t^2 e^t \, dt$.

Step 1: $u = t^2$ $\quad dv = e^t \, dt$

$du = 2t \, dt$ $\quad v = e^t$.

Step 2: $\int t^2 e^t \, dt = uv - \int v \, du$

$$= t^2 e^t - 2\int t e^t \, dt$$

Integral still complicated ...

Step 3: $u = t$ $\quad dv = e^t \, dt$

$du = dt$ $\quad v = e^t$

$$\int t e^t \, dt = t e^t - \int e^t \, dt$$

$$= t e^t - e^t + C$$

Step 4: $\int t^2 e^t \, dt = t^2 e^t - 2\left[t e^t - e^t\right] + C$

$$= t^2 e^t - 2t e^t + 2e^t + C.$$

Example: Find $\int e^x \sin x \, dx$.

Step 1: $u = e^x$ $\quad dv = \sin x \, dx$

$du = e^x \, dx$ $\quad v = -\cos x$

Step 2: $\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$

Step 3: $u = e^x$ $\quad dv = \cos x \, dx$

$du = e^x \, dx$ $\quad v = \sin x$

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx.$$

add $+\int e^x \sin x \, dx$ to both sides.

$$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x}{2} + C.$$