Today's topic: Fundamental Theorem of Calculus

Goal: establish a connection between
differential and integration

taking derivatives <=> tangent problem.

integration <=> area problem

The first part of the Fundamental Theorem deals with functions of the form:

$$\boxed{g(x) = \int_a^x f(t)\,dt}$$

where:

$f(t)$ is a continuous function $[a, b]$

x is a variable in $[a, b]$.

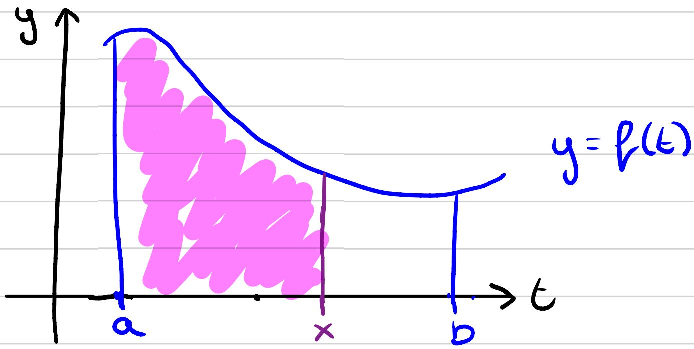
Note: $g(x)$ only depends on x.

- If x takes some value, e.g $x = 7$, then $\int_a^7 f(t)\,dt$ is a number.

think: area under the curve between $t = a$ and $t = x = 7$

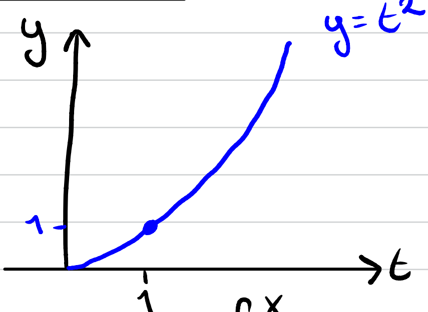
- If x varies, then the value of $\int_a^x f(t)\,dt$ also varies.

We can picture this as follows:



We can think of $g(x)$ as calculating the area under the curve "so far".

Example: Consider $g(x) = \int_1^x t^2\,dt$



Here, $f(t) = t^2$.

We can explicity find a formula for $g(x)$:

$$g(x) = \int_1^x t^2\,dt = \left[\frac{t^3}{3}\right]_1^x = \frac{x^3}{3} - \frac{1}{3}$$

Notice that since $g(x) = \frac{x^3}{3} - \frac{1}{3}$,

we have $g'(x) = \frac{\cancel{3}x^2}{\cancel{3}} = x^2$.

In this case, $g'(x) = f(x)$.

In fact, this is always true.

Fundamental Theorem of Calculus 1:

If f is a continuous function on $[a,b]$, then the function

$$g(x) = \int_a^x f(t)\,dt \qquad \text{for } a \leq x \leq b$$

is an antiderivative of f, that is

$$g'(x) = f(x).$$

In other notation: $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

Roughly: integrate first, and then differentiate, we get back the original function f.

Example : Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

- $f(t) = \sqrt{1+t^2}$ is continuous , so we can use FTC :

$$g'(x) = f(x)$$
$$= \sqrt{1+x^2}.$$

Example: Find the derivative of $g(x) = \int_1^{x^4} \sec(t)\,dt$.

- $f(t) = \sec(t)$. is continuous

Careful! Have upper bound : x^4 not just x!

- Use a substitution, partnered with Chain rule:

$$u = x^4$$

$$\frac{d}{dx}\int_1^{x^4} \sec(t)\,dt = \frac{d}{dx}\int_1^{u} \sec(t)\,dt$$

$$= \frac{d}{du}\left(\int_1^u \sec(t)\,dt\right) \cdot \frac{du}{dx}.$$

Chain rule

FTC:

$$= (\sec(u)) \cdot 4x^3$$

$$= \sec(x^4) \cdot 4x^3.$$

We will now state the FTC in full:

Fundamental Theorem of Calculus:

Let f be continuous function on $[a,b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is an antiderivative of f, ie $F' = f$.

This is just evaluation theorem.

Applications: • finding area
volume
arc length of curves
probability

etc...