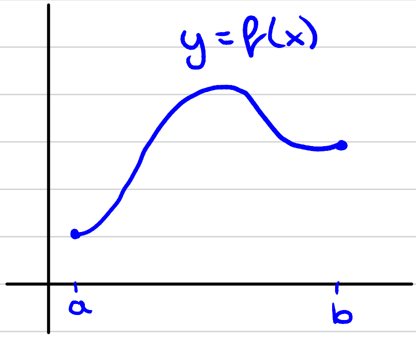
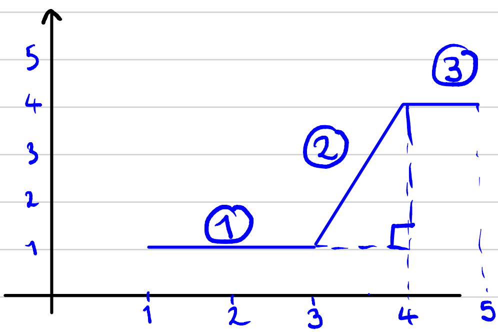
Topic: Arc Length



Imagine this curve is a string - to measure the length, we could straighten it out and measure!

That's called: arc length.
But how can we actually calculate the length?

If the curve is made up of straight line segments, we can use the distance formula on each segment:



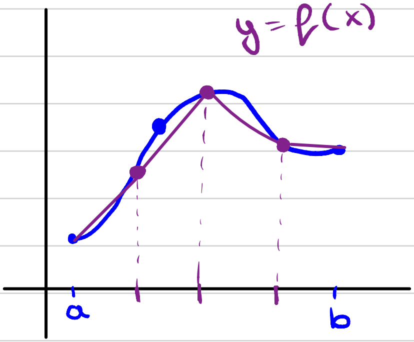
length ① $= 3 - 1 = 2$

length ② $= \sqrt{1^2 + 3^2}$
$= \sqrt{1 + 9}$
$= \sqrt{10}$.

length ③ $= 1$

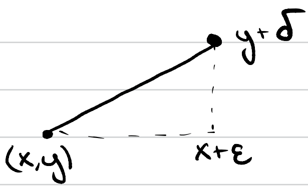
Total length $= 2 + \sqrt{10} + 1 = 3 + \sqrt{10}$.

Idea: approx our curve by line segments, add all the distances.



- make smaller line segments

Let's zoom in:

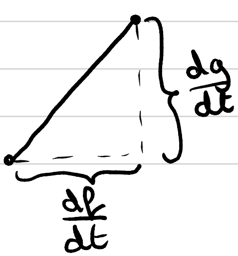


$$\text{Length} = \sqrt{\varepsilon^2 + \delta^2}$$

Suppose $x = f(t)$; $y = g(t)$

change in $x = \frac{df}{dt}$ change in $y = \frac{dg}{dt}$

$$\text{Length} = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}$$

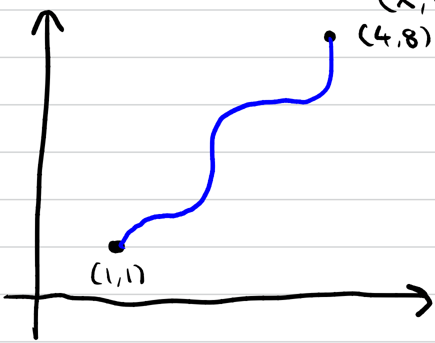


$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Arc length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example: Find the arc length of the curve given by $x = t^2$, $y = t^3$ between the points $(x,y) = (1,1)$ and $(x,y) = (4,8)$.



$x = 1$, $t = \pm 1 \Rightarrow t = 1$ $y = 1$. (if $t = -1$, $y = -1$)

$x = 4$, $\Rightarrow t = \pm 2$

because $y = 8$, $t = 2$

$1 \leq t \leq 2$.

$$L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$x = t^2$ $\quad$ $y = t^3$

$\frac{dx}{dt} = 2t$ $\quad$ $\frac{dy}{dt} = 3t^2$

$$= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt$$

$$= \int_1^2 t\sqrt{4 + 9t^2}\, dt$$

$u = 4 + 9t^2$

$du = 18t\, dt$

$\frac{1}{18} du = t\, dt$

$$= \int_{13}^{40} \frac{1}{18} \sqrt{u}\, du$$

$t = 1$ $\quad$ $u = 4 + 9 = 13$

$t = 2$ $\quad$ $u = 4 + 9.4 = 40$

$$= \frac{1}{18}\left[\frac{2u^{3/2}}{3}\right]_{13}^{40}$$

$$= \frac{1}{27}\left[80\sqrt{10} - 13\sqrt{13}\right].$$

Recall: $x = f(t)$ $\quad y = g(t)$

then arc length of the curve between $a \leq t \leq b$

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Today: $y = f(x)$ $\quad a \leq x \leq b$

Our parametrisation: $x = x$ $\quad y = f(x)$

$\frac{dx}{dx} = 1$ $\quad \frac{dy}{dx} = f'$

$$L = \int_a^b \sqrt{1 + (f')^2}\, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example: Find the arc length of the curve

$$y = \frac{x^2}{4} - \frac{1}{2}\ln(x)$$

between the points $x = 1$ and $x = 2$.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$$

$$= \frac{x^2 - 1}{2x}.$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^4 - 2x^2 + 1}{4x^2}$$

$$= \frac{4x^2}{4x^2} + \frac{x^4 - 2x^2 + 1}{4x^2}$$

$$= \frac{x^4 + 2x^2 + 1}{4x^2}$$

$$= \frac{(x^2 + 1)^2}{(2x)^2}.$$

$$L = \int_1^2 \sqrt{\frac{(x^2+1)^2}{(2x)^2}}\, dx$$

$$= \int_1^2 \frac{x^2+1}{2x} \, dx$$

$$= \int_1^2 \frac{x^2}{2x} + \frac{1}{2x} \, dx$$

$$= \int_1^2 \frac{x}{2} + \frac{1}{2x} \, dx$$

$$= \left[\frac{x^2}{4} + \frac{1}{2} \ln(x) \right]_1^2$$

$$= 1 + \frac{1}{2}\ln(2) - \left(\frac{1}{4} + \frac{1}{2} \underbrace{\ln(1)}_{=0} \right).$$

$$= \frac{3}{4} + \frac{1}{2}\ln(2).$$