Exercises

(A) $$\int \frac{4x-2}{x^2-x+2}\,dx$$

This is an indefinite integral - DO NOT FORGET + C

Since the top is linear and the bottom is quadratic, the top "resembles" the derivative of the bottom. This should scream "try u-sub"

Let $u = x^2 - x + 2$

$du = (2x-1)\,dx$

$dx = \frac{1}{2x-1}\,du$

$$\int \frac{4x-2}{u} \cdot \frac{1}{2x-1}\,du$$

$$\int \frac{2\cancel{(2x-1)}}{u} \cdot \frac{1}{\cancel{2x-1}}\,du$$

$$\int \frac{2}{u}\,du = 2 \cdot \int \frac{1}{u}\,du = 2 \cdot \ln(u)$$

undo the u-substitution

$$\boxed{= 2\ln(x^2-x+2) + C}$$

$$\int_1^5 \frac{dx}{x} = \ln 5$$

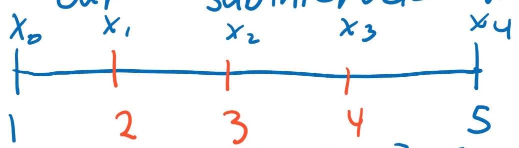
Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

For our problem $n = 4$, $a = 1$, $b = 5$, $f(x) = \frac{1}{x}$
so we have

$$\Delta x = \frac{5-1}{4} = \frac{4}{4} = 1$$

So our subintervals will be



$[1,2]$, $[2,3]$, $[3,4]$, $[4,5]$

$$\int_1^5 \frac{dx}{x} \approx \frac{1}{3}\left[\frac{1}{1} + 4\frac{1}{2} + 2\frac{1}{3} + 4\frac{1}{4} + \frac{1}{5}\right]$$

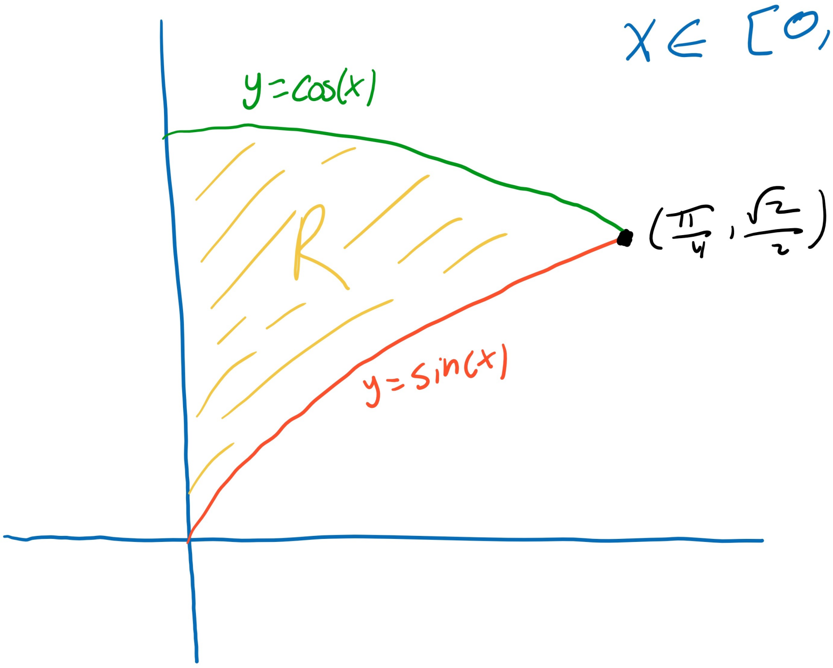
$$\int_1^5 \frac{dx}{x} \approx \frac{1}{3}\left[1 + 2 + \frac{2}{3} + 1 + \frac{1}{5}\right]$$

$$\approx \frac{1}{3}\left[\frac{73}{15}\right] = \frac{73}{45} \approx 1.6\overline{2}$$

So $\int_1^5 \frac{dx}{x} \approx \frac{73}{45}$ with Simpson's rule

Part A

$x \in [0, \frac{\pi}{4}]$



Find the Area of R

$A = \int f \text{ top} - \int f \text{ bottom}$

$A = \int_0^{\pi/4} \cos(x)\,dx - \int_0^{\frac{\pi}{4}} \sin(x)\,dx$

$A = \sin(x)\Big|_0^{\pi/4} + \cos(x)\Big|_0^{\frac{\pi}{4}} = \left(\sin(\tfrac{\pi}{4}) - \sin(0)\right)$

$+ \left(\cos\left(\tfrac{\pi}{4}\right) - \cos(0)\right)$

$A = \frac{\sqrt{2}}{2} - 0 + \frac{\sqrt{2}}{2} - 1$

$\boxed{A = \sqrt{2} - 1 \text{ units squared}}$

Part B

Find the volume when R is revolved about the y-axis using cylindrical shells

Volume formula: $V = \int_a^b 2\pi x \underbrace{r(x)}_{\text{radius}} \underbrace{h(x)}_{\text{Height}} \, dx$

Recall: when our shells are parallel to the x axis, everything in the integral should be in terms of x

$$V = \int_0^{\pi/4} 2\pi x \left(\cos(x) - \sin(x)\right) dx = 2\pi \int_0^{\pi/4} x\cos(x) - x\sin(x) \, dx$$

$$= 2\pi \int_0^{\pi/4} x\cos x \, dx - 2\pi \int_0^{\pi/4} x\sin(x) \, dx$$

Integrate by Parts

$\int x \cos x \, dx$

$u = x \quad dv = \cos(x)$

$du = 1 \quad v = \sin(x)$

$x\sin(x) - \underbrace{\int \sin(x) \, dx}_{-\cos(x)}$

$= x\sin(x) + \cos(x)$

Integrate by Parts

$u = x \quad dv = \sin(x)$

$du = 1 \quad v = -\cos(x)$

$= -x\cos(x) - \underbrace{\int -\cos(x) \, dx}_{\sin(x)}$

$= -x\cos x + \sin(x)$

Plug in your integration by parts

$$2\pi \left(x\sin(x) + \cos(x)\right) - 2\pi \left(-x\cos(x) + \sin(x)\right)$$

$$= 2\pi \left[x\sin(x) - \sin(x) + x\cos(x) + \cos(x)\right] \Big|_0^{\pi/4}$$

$$\boxed{= \frac{\pi(2\sqrt{\pi} - 4)}{2} \text{ units cubed}}$$

Riemann Sum Practice

The area under a curve is approximated by

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

This reads as $f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$

So if the number of boxes gets larger, the area gets more accurate

$$A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* = a + \Delta x \cdot i$

Polar coordinates

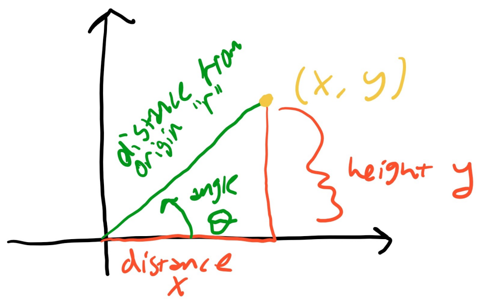
Every point (x, y) can be described by the distance from the origin

$$r = \sqrt{x^2 + y^2}$$

and its angle with the x-axis

$$\theta = \arctan\left(\frac{y}{x}\right)$$

These coordinates are called polar coordinates



F³ The polar coord. $(-r, \theta) = (r, \theta \pm \pi)$

So we have the following relationships

$\cos(\theta) = \frac{x}{r}$ $\quad$ $\sin(\theta) = \frac{y}{r}$

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

Polar curves have the form

$r = f(\theta)$

The curve is the set of all (r, θ) such that $r = f(\theta)$

Tangents to polar curves

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cos\theta)}$$

because $r = f(\theta)$

$x = r\cos\theta = f(\theta)\cos(\theta)$

$y = r\sin\theta = f(\theta)\sin(\theta)$

So $$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Area with polar Coordinates

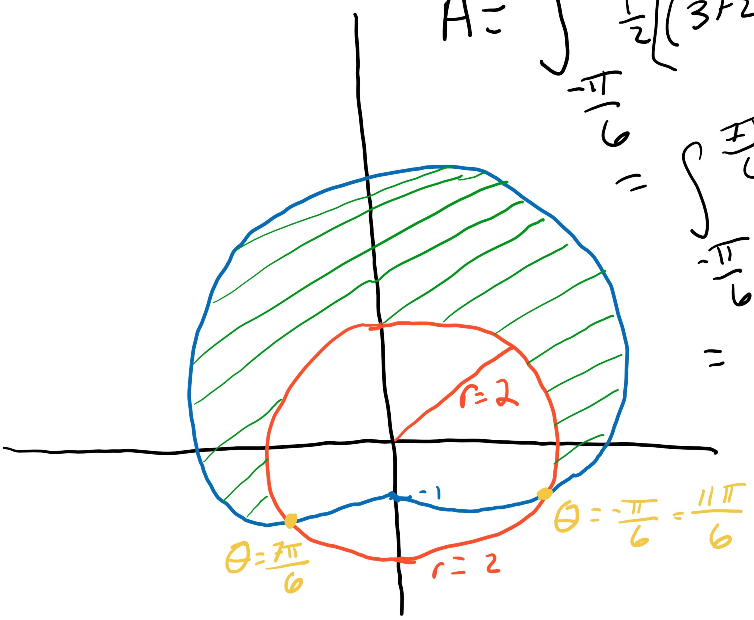
Determine the area inside $r = 3 + 2\sin\theta$ and outside $r = 2$

$$A = \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left[(3+2\sin\theta)^2 - 2^2\right] d\theta$$

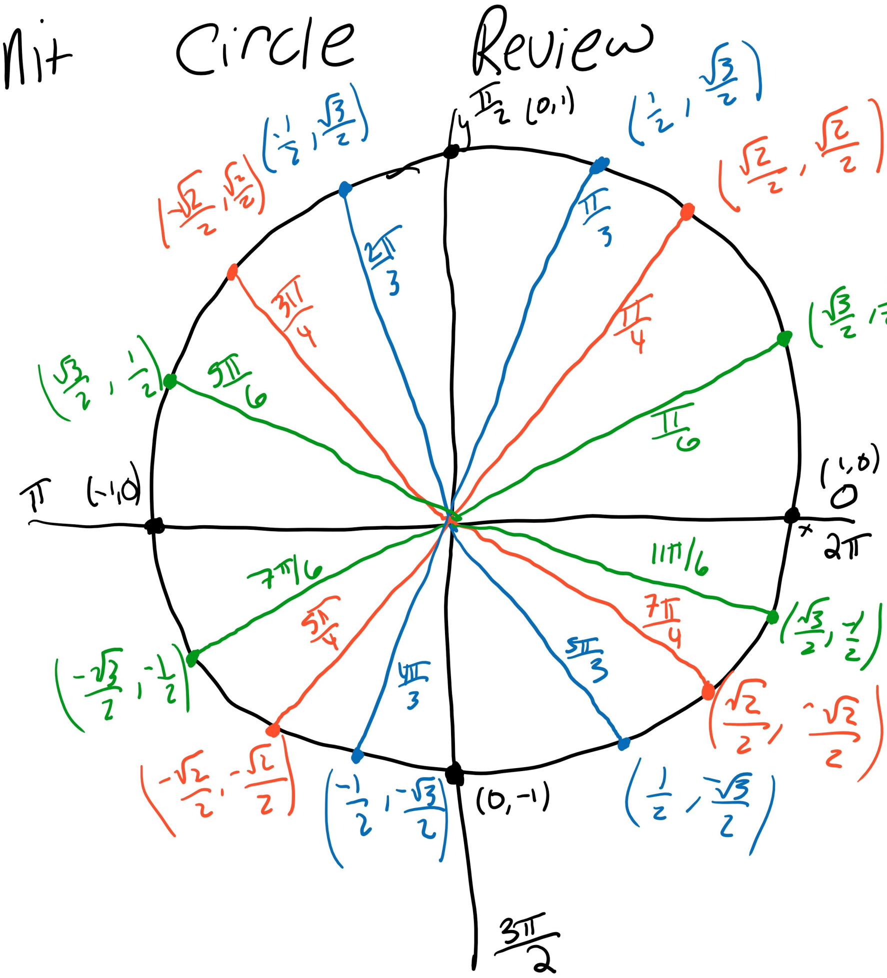
$$= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left(7 + 12\sin\theta - 2\cos(2\theta)\right)$$

$$= \frac{1}{2}\left(7\theta - 12\cos\theta - \sin(2\theta)\right)\Big|_{-\pi/6}^{7\pi/6}$$

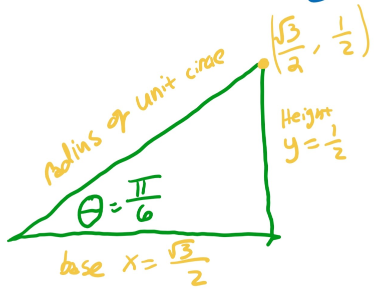
$$\boxed{= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3}}$$



Unit Circle Review



Let's look at $\theta = \frac{\pi}{6}$



U-substitution

$$\int \sin(204x)\, dx = \frac{-1}{204}\cos(204x) + C$$

let
$$u = 204x$$
$$du = 204\, dx$$
$$dx = \frac{1}{204}\, du$$

$$\int \sin(u) \cdot \frac{1}{204}\, du = \frac{1}{204}\int \sin(u)\, du = \frac{1}{204}\left(-\cos(u)\right) + C$$

Arclength Problem

Find the arclength of $y = \ln(\sec(x))$ on the interval $0 \le x \le \frac{\pi}{4}$

$L = \int ds$ and $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ if $y = f(x)$
$a \le x \le b$

$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$ if $x = g(y)$ and $c \le y \le d$

Compute ds for $y = f(x) = \ln(\sec(x))$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(\sec(x))\right) = \tan(x)$$

$$\left(\frac{dy}{dx}\right)^2 = \tan^2(x)$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)}$$

Recall: $\sec^2(x) = 1 + \tan^2(x)$ because $\sin^2(x) + \cos^2(x) = 1$

$$\frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$1 + \tan^2(x) = \sec^2(x)$$

so $L = \int_0^{\pi/4} \sqrt{1 + \tan^2(x)} = \int_0^{\pi/4} \sqrt{\sec^2(x)} = \int_0^{\pi/4} \sec(x)\,dx$

$$= \ln\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right) - \ln(\sec(0) + \tan(0))$$

$= \ln(\sqrt{2} + 1)$ is our arclength

Trig Substitutions

If you see

$\sqrt{a^2+x^2}$ think $x = a\tan(\theta)$

$\sqrt{a^2-x^2}$ think $x = a\sin(\theta)$

$\sqrt{x^2-a^2}$ think $x = a\sec(\theta)$

Improper Integrals

$$\int_a^\infty f(x)\,dx = \lim_{n\to\infty} \int_a^n f(x)\,dx$$

$$\int_{-\infty}^b g(x)\,dx = \lim_{n\to\infty} \int_n^b g(x)\,dx$$

$$\int_{-\infty}^\infty h(x)\,dx = \lim_{n\to\infty} \int_0^n h(x)\,dx + \lim_{n\to\infty} \int_n^0 h(x)\,dx$$

Example

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x^2} = \lim_{n\to\infty} -\frac{1}{x}\Big|_1^n$$

$$\lim_{n\to\infty} \left(\frac{-1}{n} + 1\right) = 0 + 1 = 1 \quad \text{so} \quad \int_1^\infty \frac{1}{x^2} = 1$$