MAT 126-Exam 2-Spring 2018

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*Each numbered question is worth 20 points.

1. For all parts in question #1 $f'(x) = x\sqrt{9-x^2}$

a.) Find a general formula for f.

$$\int x\sqrt{9-x^2}\,dx$$

LET $u = 9-x^2$
$du = -2x\,dx$

$$= -\frac{1}{2}\int \sqrt{u}\,du$$

$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$$

$$= \boxed{-\frac{1}{3}(9-x^2)^{3/2} + C}$$

b) Find the exact area under f' from $x = 0$ to $x = 3$.

$$\int_0^3 x\sqrt{9-x^2} = -\frac{1}{3}(9-x^2)^{3/2}\Big|_0^3$$

$$= -\frac{1}{3}\left(0^{3/2} - 9^{3/2}\right) = \frac{3^3}{3} = \boxed{9}$$

c) Find a formula for f if $f(-3) = 2$ ONE CORRECT SOLUTION IS $2+\int_{-3}^{x} f(t)\,dt$

BUT SINCE $f(x) = -\frac{1}{3}(9-x^2)^{3/2} + C$

WE HAVE $f(-3) = -\frac{1}{3}(9-9)^{3/2} + C = C$

SO $C = 2$

$$\boxed{f(x) = -\frac{1}{3}(9-x^2)^{3/2} + 2}$$

d) Use integration by ~~a~~ substitution to find an antiderivative of f or show why this is not possible.

WE WANT $-\frac{1}{3}\int (9-x^2)^{3/2} + 2\,dx = -\frac{2x}{3} - \frac{1}{3}\int (9-x^2)^{3/2}\,dx$

$\boxed{\text{LET } x = 3\sin\theta \quad dx = 3\cos\theta\,d\theta}$ TO GET $-\frac{2x}{3} - \frac{1}{3}\int (9-9\sin^2\theta)^{3/2}\cdot 3\cos\theta\,d\theta = -\frac{2x}{3} - 9^{3/2}\int (1-\sin^2\theta)^{3/2}\cos\theta\,d\theta$

USE $1-\sin^2\theta = \cos^2\theta$ → $= -\frac{2x}{3} - 27\int \cos^4\theta\,d\theta$

USE $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

SO $\cos^4\theta = \left[\frac{1}{2}(\cos 2\theta + 1)\right]^2$

$= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$

$= \frac{1}{4}\left(\frac{1}{2}\cos 4\theta + \frac{1}{2} + 2\cos 2\theta + 1\right)$

$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

$$= -\frac{2x}{3} - \frac{27}{8}\int \cos 4\theta + 4\cos 2\theta + 3\,d\theta$$

$$= -\frac{2x}{3} - \frac{27}{8}\left(\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right) + C$$

USE $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

TO GET

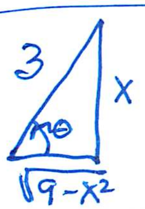
$\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$= 4\sin\theta\cos\theta(1-2\sin^2\theta)$

$$= -\frac{2x}{3} - \frac{27}{8}\left(\sin\theta\cos\theta(1-2\sin^2\theta) + 4\sin\theta\cos\theta + 3\theta\right) + C$$

$$= -\frac{2x}{3} - \frac{27}{8}\left(\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\left(1-\frac{2x^2}{9}\right) + \frac{4x}{3}\frac{\sqrt{9-x^2}}{3} + 3\arcsin\left(\frac{x}{3}\right)\right)$$

$$= -\frac{2x}{3} - \frac{27}{8}\left(\frac{3x-2x^2}{27}\sqrt{9-x^2} + \frac{4x}{9}\sqrt{9-x^2} + 3\arcsin\left(\frac{x}{3}\right)\right) + C$$



SINCE $\sin\theta = x/3$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

BLEAH!

SPR 18 SOL'NS

5.Find the exact value of the following:

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx$$

WE CAN DO THIS TWO WAYS:

① FIRST, NOTICE THAT

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^4}\,dx + \int_{0}^{\infty} \frac{x}{1+x^4}\,dx$$

$\boxed{\text{IF}}$ IT CONVERGES.

$\left(\begin{array}{l}\text{LET } u=-x \\ du=-dx\end{array}\right)$ $$= \int_{0}^{\infty} \frac{-u}{1+u^4}\,du + \int_{0}^{\infty} \frac{x}{1+x^4}\,dx = 0$$

THE INTEGRAL $\int_0^\infty \frac{x}{1+x^4}$ CONVERGES, SINCE

$$0 < \frac{x}{1+x^4} < \frac{x}{x^4} = \frac{1}{x^3}$$

AND $\int_1^\infty \frac{dx}{x^3}$ CONVERGES.

SO IT IS $\boxed{0}$

② ALTERNATIVELY,

$$\int_0^\infty \frac{x}{1+x^4}\,dx = \frac{1}{2}\int_0^\infty \frac{1}{1+u^2}\,du = \lim_{M\to\infty} \arctan M - \arctan 0$$

$\boxed{\begin{array}{ll}\text{LET } u=x^2 & x=0 \Rightarrow u=0 \\ du=2x\,dx & x=\infty \Rightarrow u=\infty\end{array}}$ $= \pi/2$.

SIMILARLY $$\int_{-\infty}^{0} \frac{x}{1+x^2}\,dx = -\frac{\pi}{2}$$

$$\boxed{-\frac{\pi}{2} + \frac{\pi}{2} = 0}$$

SPR 18 SOL'NS

(ALL 3 DIVERGE SINCE $\int_1^\infty \frac{dx}{x}$ AND $\int_0^1 \frac{dx}{x}$ DO)

4) Compute the following or show divergence for $f(x) = \frac{1}{x}$

a) $\int_{-\infty}^{-2} f(x)dx = \lim_{N\to-\infty} \int_N^{-2} \frac{dx}{x}$

$= \lim_{N\to-\infty} \ln|x| \Big|_{-\infty}^{-2} = \lim_{N\to-\infty} \ln|N| - \ln 2$ $\boxed{\text{DIVERGES}}$

SINCE $\lim_{x\to+\infty} \ln x = +\infty$

b) $\int_1^0 f(x)dx = -\int_0^1 \frac{1}{x}dx$

$= \lim_{a\to0^+} \int_a^1 \frac{dx}{x} = \lim_{a\to0^+} \ln|x| \Big|_a^1 = \lim_{a\to0^+} \ln a - \ln 1$

$= \lim_{a\to0^+} \ln a = -\infty$

$\boxed{\text{DIVERGES}}$

c) $\int_{-3}^{3} f(x)dx = \int_{-3}^{0} \frac{1}{x}dx + \int_0^3 \frac{1}{x}dx$

$= \lim_{b\to0^-} \ln|x| \Big|_{-3}^{b} + \lim_{a\to0^+} \ln|x| \Big|_a^3$

$= \left(\lim_{b\to0^-} \ln|b| - \ln 3\right) + \left(\lim_{a\to0^+} \ln 3 - \ln|a|\right)$

$= -\infty - \ln 3 + (\ln 3 + \infty)$ $\boxed{\text{DIVERGES}}$

SINCE BOTH PIECES DO

DOES NOT CANCEL.

SPR'18 SOL'NS

3. Find all antiderivatives of $y = \tan 2x + x\sqrt{x-1} - e^{-x}$

$$\int \tan 2x + x\sqrt{x-1} + e^{-x}\, dx$$

$$= \int \frac{\sin 2x}{\cos 2x}\, dx + \int x\sqrt{x-1}\, dx + \int e^{-x} dx$$

$u = \cos 2x$

$du = -2\sin 2x\, dx$

$w = x-1$

$dw = dx$

$w+1 = x$

$$= -\frac{1}{2}\int \frac{du}{u} + \int (w+1)\, w^{1/2} dw - e^{-x} + C$$

$$= -\frac{1}{2}\ln|\cos 2x| + \int w^{3/2} + w^{1/2} dw - e^{-x} + C$$

$$= \boxed{-\frac{1}{2}\ln|\cos 2x| + \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} - e^{-x} + C}$$

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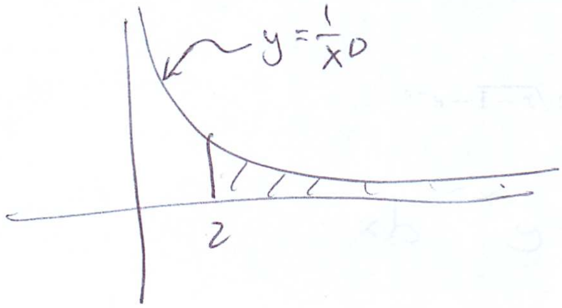
2. For which values of p does $y = \frac{1}{x^p}$ have a finite area under the curve for $x \geq 2$? Prove your answer.



$$\int_2^\infty \frac{dx}{x^p} = \lim_{M \to \infty} \int_2^M \frac{dx}{x^p} = \lim_{M \to \infty} -\frac{x^{-p+1}}{1-p} \Big|_2^M$$

$$= \lim_{M \to \infty} \frac{1}{1-p}\left(-M^{1-p} + 2^{1-p}\right)$$

IF $p > 1$, THEN $1-p < 0$, SO $\lim_{M \to \infty} \left(M^{1-p}\right) = 0$

BUT, IF $p < 1$, $1 - p \geqslant 0$ AND THE LIMIT DIVERGES.

THUS, IF $\boxed{p > 1}$, THE AREA IS FINITE SINCE THE INTEGRAL EXISTS.