MAT 126-Exam 1-Spring 2018

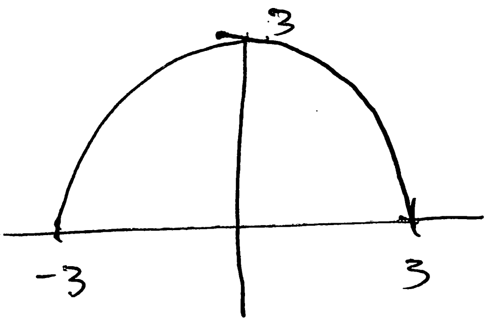
NAME:___SOLUTIONS___

TA NAME:______________________________

*Each numbered question is worth 20 points.

1. For all parts in question #1 $f'(x) = \sqrt{9 - x^2}$

a.) Sketch a graph of f'.



(THIS IS THE UPPER HALF OF THE CIRCLE $x^2 + y^2 = 9$)

b) Write an expression in sigma notation that represents the area under f' from $x = 0$ to $x = 3$.

$$\boxed{\lim_{n\to\infty} \frac{3}{n} \sum \sqrt{9 - \left(\frac{3i}{n}\right)^2}} = \int_0^3 \sqrt{9 - x^2}\, dx$$

(HERE $a = 0$, $b = 3$, SO $\Delta x = \frac{3}{n}$ AND $x_i = 0 + \frac{3}{n} i$ WITH $f(x) = \sqrt{9 - x^2}$)

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c) Find the exact value of the area under f' from $x = 0$ to $x = 3$.

SINCE THIS IS $\frac{1}{4}$ OF A CIRCLE OF RADIUS 3,

THE AREA IS $\frac{1}{4}(\pi \cdot 3^2) = \boxed{\frac{9}{4}\pi}$

d) Sketch a graph of f if $f(0) = 5$.

THE DOMAIN OF f IS -3 TO 3

$f'(-3) = f'(3) = 0$ (SINCE $f'(x) = \sqrt{9-x^2}$)

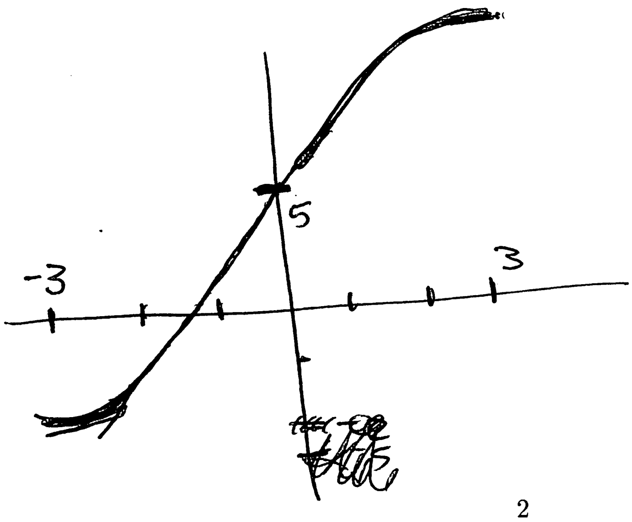
$f'(x) > 0$ FOR $|x| < 3$. $f'(0) = 3$.

$f(x) = k + \int_{-3}^{x} \sqrt{9-t^2}\,dt$, WHERE k IS CHOSEN SO $f(0) = 5$.

AT $x = 0$, WE KNOW $5 = f(0) = k + \int_{-3}^{0} \sqrt{9-t^2}\,dt$

$= k + \frac{9\pi}{4} \approx k + 7$ $\left(\frac{9\pi}{4} \approx 7\right)$

SO k IS ABOUT -2.



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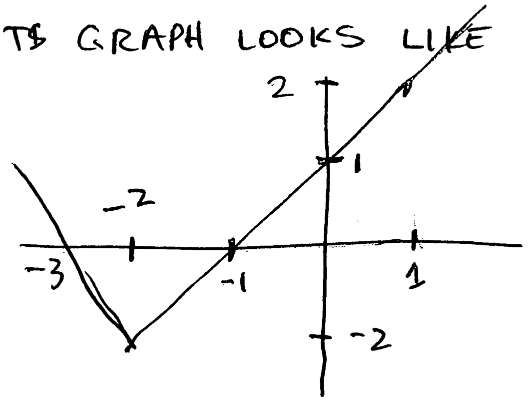
2. Draw $y = F(x) = \int_1^x (-1 + |t+2|)dt$ with correct concavity on a scaled set of axes. (Include at least 3 labeled points.)

FIRST, LETS THINK ABOUT $F'(x) = -1 + |x+2|$.

ITS GRAPH LOOKS LIKE



$F(x) = \int_1^x F'(t)\,dt$

SO

$F(1) = 0$

(SINCE IT IS AN INTEGRAL FROM 1 TO 1)

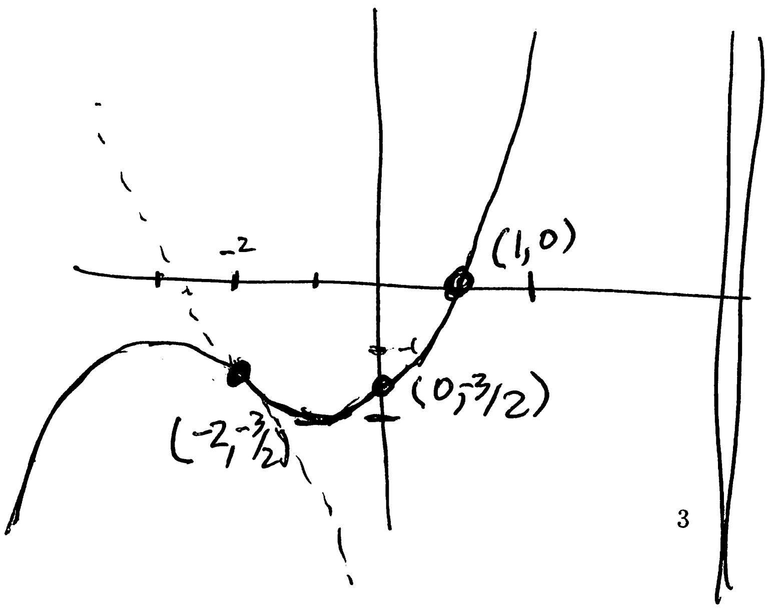
AND FOR $x > -2$, $F'(x)$ IS JUST THE LINE $y = x+1$,

SO $F(x)$ WILL BE A PARABOLA,

WITH VERTEX @ -1 : $\frac{x^2}{2} + x - \frac{3}{2}$ FOR $x > 2$

$F(-1) = -2$ (SINCE IT IS THE AREA OF A TRIANGLE WITH BASE 2, HT 2)

$F(-2) = -\frac{3}{2}$



WHEN $x < -2$,

$F'(x) = -x - 3$,

BUT WE NEED TO CHOOSE THE CONSTANT SO IT IS CONTINUOUS AT $x = -2$

SO

$F(x) = -\frac{x^2}{2} - 3x + \frac{7}{2}$ FOR $x < 2$

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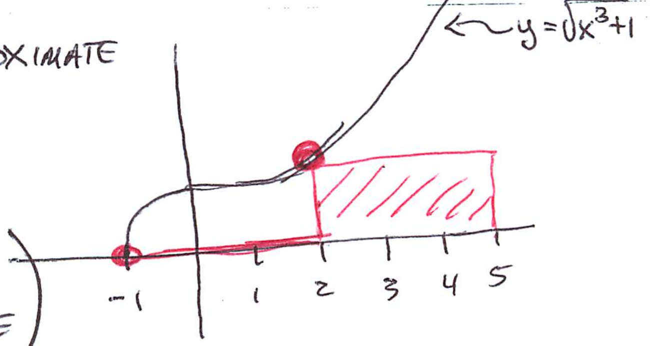
3. Use a left Riemann estimate with 2 subintervals to approximate the area between $\frac{dy}{dx} = \sqrt{x^3+1}$ and the x axis from $x=-1$ to $x=5$. Now use this value to sketch $y=f(x)$ if $f(-1)=2$

FOR THE FIRST PART, WE APPROXIMATE $\int_{-1}^{5} \sqrt{x^3+1}\,dx$ WITH TWO RECTANGLES, EVAL. ON LEFT (THE GRAPH IS AT RIGHT, BUT YOU DON'T NEED THIS TO DO THE PROBLEM)



SINCE $a=-1$, $b=5$, $\Delta x = \frac{b-a}{n} = \frac{6}{2} = 3$, SO $x_0=-1$, $x_1=2$

LET $g(x) = \sqrt{x^3+1}$, AND WE HAVE

$$\int_{-1}^{5} \sqrt{x^3+1}\,dx \approx 3\sum_{i=0}^{1} g(x_i) = 3\left(g(-1)+g(2)\right) = 3\left(\sqrt{(-1)^3+1} + \sqrt{2^3+1}\right) = 3(0+\sqrt{9}) \boxed{=9}$$

NOW LET $f(x)$ BE SO THAT $f'(x) = \sqrt{x^3+1}$ WITH $f(-1)=2$.

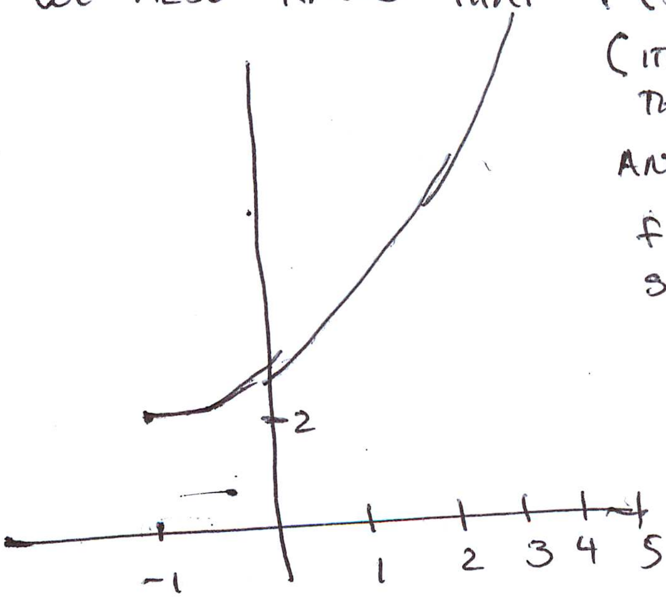
THIS MEANS $f(x) = 2 + \int_{-1}^{x} \sqrt{t^3+1}\,dt$ (WHICH WE CAN'T WRITE A FORMULA FOR)

WE KNOW $f(-1)=2$, $f'(-1)=0$, $f'(0)=1$, AND $f'(x)>0$ FOR $x>-1$

WE ALSO KNOW THAT $f(5) \approx 9+2=11$

(ITS ACTUALLY CLOSER TO 27, BUT THATS NOT PART OF THE PROBLEM)

AND, IF YOU'RE REALLY INTO IT $f''(x) = \frac{3}{2}x^2(x^3+1)^{-1/2} > 0$ FOR ALL X, SO f IS CONCAVE UP)



AT LEFT IS A SKETCH OF $f(x) = 2 + \int_{-1}^{x} \sqrt{t^3+1}\,dt$

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4) Compute the following for $f(x) = \sin x + 2x$

a) $\int_0^{2\pi} f(x)dx$

$$\int_0^{2\pi} \sin x + 2x \, dx = -\cos x + x^2 \Big|_0^{2\pi} = -1 + (2\pi)^2 - (-1 + 0^2)$$
$$= (2\pi)^2 = 4\pi^2$$

b) $\int_{2\pi}^{0} f(x)dx$

$$\int_{2\pi}^{0} f(x)dx = -\int_0^{2\pi} f(x)dx = -4\pi^2$$

c) $\lim_{n\to\infty} \frac{4\pi}{n} \sum_{i=1}^{n} f(x_i)$

THIS QUESTION DOESN'T EVEN MAKE SENSE UNLESS SOMETHING IS SAID ABOUT WHAT X_i IS. LETS ASSUME $X_i = \frac{2\pi}{n} i$.

THEN

$$\lim_{n\to\infty} \frac{4\pi}{n} \sum_{i=1}^{n} f\left(\frac{2\pi i}{n}\right) = 2 \lim_{n\to\infty} \frac{2\pi}{n} \sum f\left(\frac{2\pi i}{n}\right) = 2 \int_0^{2\pi} f(x)dx$$
$$= 2 \cdot 4\pi^2 = 8\pi^2.$$

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(TO DO THIS, WE NEED TO REMEMBER THAT

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1+4+9+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} 1 = 1+1+1+\cdots+1 = n.$$
)

5.Using a right Riemann sum, compute the following using limits:

$$\int_1^3 x^2 dx$$

[HERE $a=1, b=3$ SO $\Delta X = \frac{2}{n}$
AND $X_i = 1+\frac{2i}{n}$]

$$\int_1^3 x^2 dx = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{n}\sum_{i=1}^{n} 1 + \frac{4}{n}\sum_{i=1}^{n} i + \frac{4}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{n}(n) + \frac{8}{n^2}\cdot\frac{(n+1)n}{2} + \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 2 + \lim_{n\to\infty}\frac{8(n^2+n)}{2n^2} + \lim_{n\to\infty}\frac{8(2n^3+3n^2+n)}{6\,n^3}$$

$$= 2 + 4 + \frac{8}{3} = \frac{26}{3}$$

(AS A CHECK,

$$\int_1^3 x^2 dx = \frac{1}{3}x^3\Big|_1^3 = \frac{1}{3}(27-1) = \frac{26}{3}.$$ SO THAT'S GOOD)