SOLUTIONS, SPRING 2014.

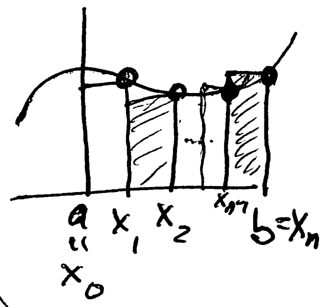
1. (*a*) Write the following integral as a limit choosing the sample points to be the midpoints:

$$\int_0^1 5\cos x\, dx.$$

Notice: your answer should not contain symbols x_i or Δx. Plug all the formulas in your answer. You don't need to compute the integral.

RECALL THAT

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$



WHERE $\Delta x = \frac{b-a}{n}$, $x_0 = a$, $x_1 = a+\Delta x$, $x_2 = a + 2\Delta x$, ..., $x_n = b = a + n\Delta x$.

(SO $x_i = a + i\Delta x$)

IN THIS CASE, $a=0$, $b=1$ SO $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

AND $x_i = \frac{i}{n}$.

WE TAKE $f(x) = 5\cos(x)$

SO WE GET

$$\boxed{\begin{aligned}\int_0^1 5\cos x\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} 5\cos\left(\frac{i}{n}\right)\cdot\frac{1}{n} \\ &= \lim_{n\to\infty} \frac{5}{n}\sum_{i=1}^{n} \cos\left(\frac{i}{n}\right)\end{aligned}}$$

EITHER IS OK.

SOLUTIONS, SPR 2014

(b) Write the following limit as a definite integral:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln(2 + \tfrac{i}{n}).$$

Is the Riemann sum $\frac{1}{n}\sum_{i=1}^{n} \ln(2 + \frac{i}{n})$ an underestimate, an overestimate of this integral or neither one?

Notice: you don't need to compute the integral. !

- HERE WE CAN TAKE $\Delta x = \frac{1}{n}$, (SEE PREV PROBLEM)

$f(x) = \ln(x)$

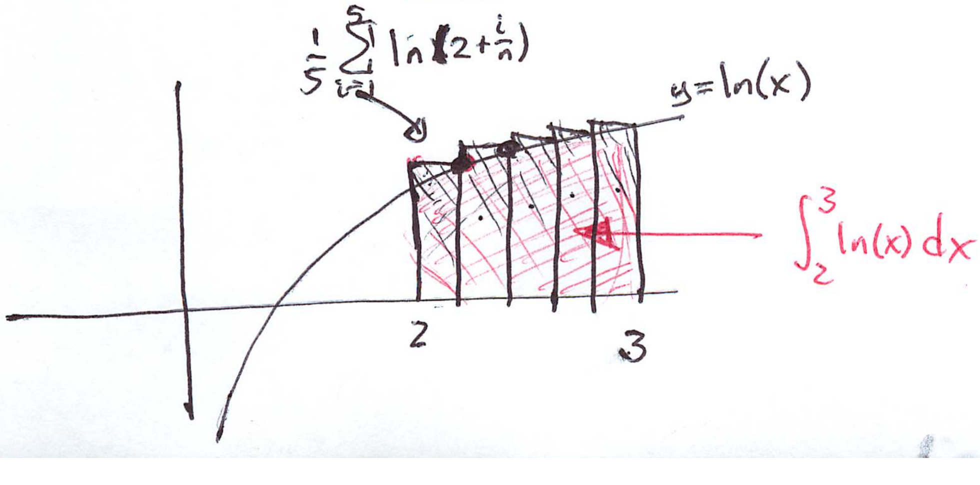
SO $x_i = 2 + \frac{i}{n} = 2 + \Delta x$.

THAT MEANS $a = 2$ AND $b = a + n\Delta x = 2 + n \cdot \frac{1}{n} = 3$.

SO $$\boxed{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln\left(2+\frac{i}{n}\right) = \int_2^3 \ln x \, dx}$$

[ALSO OK IS $\int_0^1 \ln(2+x)\,dx$ OR MANY OTHERS]

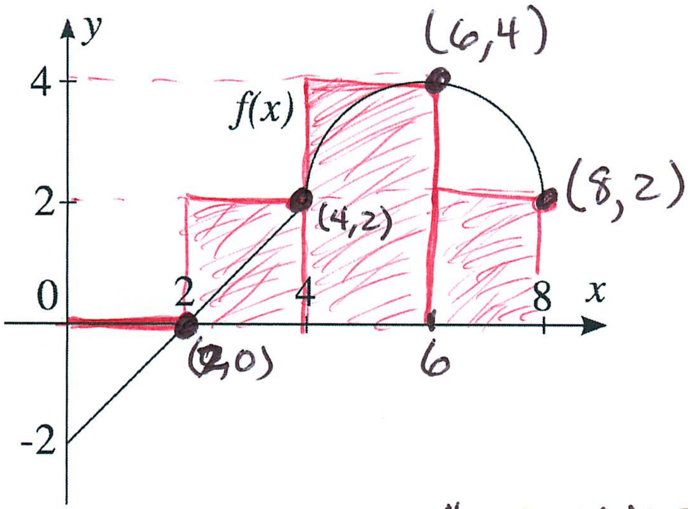
- SINCE $\ln x$ IS AN INCREASING FUNCTION AND WE'RE TAKING RIGHT ENDPOINTS, IT IS AN $\boxed{\text{OVERESTIMATE}}$



2. (*a*) Estimate the integral

$$\int_0^8 f(x)\mathrm{d}x$$

using the right endpoints with $n = 4$ for the function whose graph is shown below.



SINCE WE ARE USING 4 RECTANGLES WITH <u>RIGHT</u> ENDPOINTS, $a=0$, $b=8$, $\Delta x = 2$

WE HAVE $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, $x_4 = 8$.

SO WE COMPUTE

$$2\sum_{i=1}^{4} f(x_i) = 2\big(f(2) + f(4) + f(6) + f(8)\big)$$

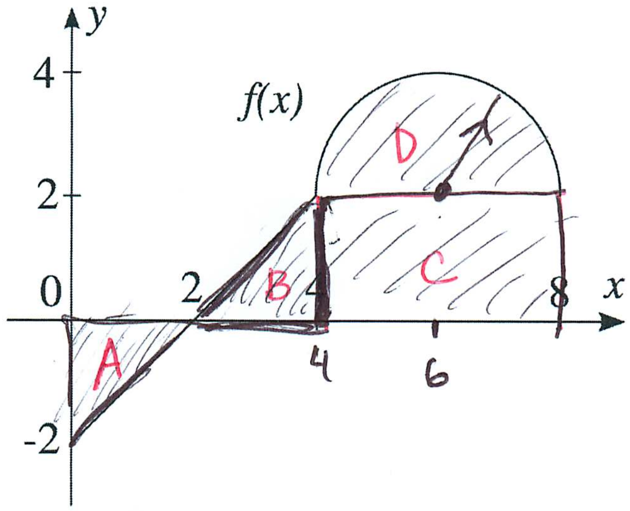
$$= 2(0 + 2 + 4 + 2)$$

$$= 2 \cdot 8 = \boxed{16}$$

(*b*) Find the exact value of

$$\int_0^8 f(x)\mathrm{d}x$$

using geometry.



THE INTEGRAL CAN BE CALCULATED AS THE SUM OF SIGNED AREAS OF THE FOUR REGIONS

A (TRIANGLE, BASE 2, HT −2)
B (TRIANGLE, BASE 2, HT +2)
C (RECTANGLE BASE 4, HT 2)
D (SEMICIRLE RADIUS 2)

SINCE A & B HAVE EQUAL AREAS BUT OPPOSITE SIGNS, THEY CANCEL OUT.

AREA (C) = 8

AREA (D) = $\frac{1}{2}\pi(2)^2 = 2\pi$.

THUS $\boxed{\int_0^8 f(x)\,dx = 8 + 2\pi}$

SOLUTIONS, SPR 2014

3. Evaluate the following definite integrals:

$$(a) \int_1^5 2u^{\frac{3}{4}}\, du; \qquad (b) \int_0^{2\pi} (3\sin t - e^t) dt.$$

a) $\int_1^5 2u^{3/4} du = 2 \cdot \frac{4}{7} u^{7/4} \Big|_1^5$

$= \frac{8}{7}\left(5^{7/4} - 1^{7/4}\right)$ ← EITHER

$= \frac{8}{7}\left(5^{7/4} - 1\right)$ ←

b) $\int_0^{2\pi} (3\sin t - e^t)\, dt$

$= -3\cos t - e^t \Big|_0^{2\pi}$

$= \left(-3\cos(2\pi) - e^{2\pi}\right) - \left(-3\cos(0) - e^0\right)$

$= \left(-3 - e^{2\pi}\right) - (-3 - 1)$

$= \boxed{1 - e^{2\pi}}$

SOLUTIONS, SPR 2014

4. The velocity function (in meters per second) for a particle moving along a line is given:

$$v(t) = -t^2 + 4.$$

Find (a) the displacement and (b) the distance traveled by the particle during the time interval $0 \leqslant t \leqslant 5$.

Notice: if $v(t)$ changes sign it means that the particle starts moving in an opposite direction. The displacement is the distance between the starting and the end points of the particle. The distance traveled is the total distance traveled by the particle in both directions.

THE DISPLACEMENT IS JUST

$$\int_0^5 (-t^2+4)\,dt = -\frac{1}{3}t^3 + 4t \Big|_0^5 = \left(\frac{-125}{3}+20\right) - (0)$$

$$= \boxed{\frac{-65}{3} \text{ METERS.}}$$

THE TOTAL DISTANCE TRAVELED IS

$$\int_0^5 |-t^2+4|\,dt,$$

WHICH WE HAVE TO SPLIT AT $t=2$, SINCE THAT IS WHERE THE PARTICLE REVERSES DIRECTION.

SO

$$\int_0^5 |-t^2+4|\,dt = \int_0^2 (-t^2+4)\,dt + \int_2^5 (t^2-4)\,dt$$

$$= \left(-\frac{t^3}{3}+4t\right)\Big|_0^2 + \left(\frac{t^3}{3}-4t\right)\Big|_2^5$$

$$= \left(-\frac{8}{3}+8\right) - 0 + \left(\frac{125}{3}-20\right) - \left(\frac{8}{3}-8\right)$$

$$= \frac{16}{3} + \frac{65}{3} - \left(-\frac{16}{3}\right) = \frac{97}{3} \text{ METERS}$$

5. Let

$$g(x) = \int_1^{x^4} te^{\frac{t}{2}}\, \mathrm{d}t$$

Compute the derivative $g'(x)$ using the Fundamental Theorem of Calculus.

$$\boxed{g'(\cancel{x}) = x^4 e^{x^4/2} \cdot 4x^3 = 4x^7 e^{x^4/2}}$$

IF YOU ARE CONFUSED BY THIS,

LET $F(u) = \int_1^u te^{t/2}\,dt$

AND THE F.T.C. SAYS THAT

$$F'(u) = ue^{u/2}$$

NOW $g(\cancel{x}) = F(x^4)$

SO $g'(x) = F'(x^4) \cdot 4x^3$ BY THE CHAIN RULE.

SO $g'(x) = (x^4 e^{x^4/2}) \cdot 4x^3$.

YOU MIGHT HAVE KNOWN THAT

$$g(x) = \int_1^{x^4} te^{t/2}\,dt = 2(t-2)e^{t/2}\Big|_1^{x^4}$$

$$= 2x^4 e^{x^4/2} - 4e^{x^4/2} + 2e^{1/2}$$

SO $g'(x) = (8x^3 e^{x^4/2} + 4x^7 e^{x^4/2}) - 8x^3 e^{x^4/2} = 4x^7 e^{x^4/2}$

BUT THAT'S A REALLY HARD WAY TO DO THIS.