Stony Brook University

NAME: *Final Exam Solutions*

RECITATION #: _____

MAT 126 – Spring 2016 – Final Exam

May 11, 2016

INSTRUCTIONS – PLEASE READ

- Please turn off your cell phone and put it away.
- Please write your name and your section number right now.
- This is a closed book exam. You are NOT allowed to use a calculator or any other electronic device or aid.
- The final has 8 problems worth a total of 150 points. Look over your test packet as soon as the exam begins. If you find any missing pages or problems please ask a proctor for another test booklet.
- Show your work. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly. You do not need to simplify your answers unless explicitly instructed to do so.
- Academic integrity is expected of all Stony Brook University students at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
Total	

LEC 01	MWF	10:00-10:53am	Joseph Adams
R01	F	1:00-1:53pm	Jaroslaw Jaracz
R02	Tu	4:00-4:53pm	Charles Cifarelli
R03	Tu	1:00-1:53pm	Jaroslaw Jaracz
R04	Th	8:30-9:23am	Alaa Abd-El-Hafez
R05	M	1:00-1:53pm	Thomas Rico
R06	M	9:00-9:53am	Zhuang Tao
R07	W	11:00-11:53am	Dyi-Shing Ou
LEC 02	TuTh	2:30-3:50pm	Raluca Tanase*
R08	Tu	4:00-4:53pm	Gaurish Telang
R09	Tu	1:00-1:53pm	Yuan Gao
R10	Th	1:00-1:53pm	Alaa Abd-El-Hafez
R11	F	1:00-1:53pm	Ruijie Yang
R12	W	12:00-12:53pm	Christopher Ianzano
R13	M	10:00-10:53am	Zhuang Tao
R14	M	12:00-12:53pm	Thomas Rico
LEC 03	MW	4:00-5:20pm	David Kahn
R15	W	9:00-9:53am	Ruijie Yang
R16	Tu	10:00-10:53am	Ying Chi
R17	W	10:00-10:53am	Ying Chi
R18	Th	4:00-4:53pm	Gaurish Telang
R31	W	5:30-6:23pm	Mariangela Ferraro
R32	M	5:30-6:23pm	Charles Cifarelli
R33	Tu	1:00-1:53pm	Yu Zeng

Some trigonometric formulas that might be useful:	
$\sin^2(x) + \cos^2(x) = 1$	$\sin(2x) = 2\sin(x)\cos(x)$
$\tan^2(x) = \sec^2(x) - 1$	$\cos(2x) = 2\cos^2(x) - 1 = \cos^2(x) - \sin^2(x)$

Problem 1. (38 points) Evaluate the following integrals:

a) $\int_{-1}^{1} 5x^3 + 3x + 1\,dx \quad = 5\frac{x^4}{4} + 3\frac{x^2}{2} + x \Big|_{-1}^{1}$

$= \left(\cancel{5\frac{1}{4}} + \cancel{3\cdot\frac{1}{2}} + 1\right) - \left(\cancel{+\frac{5}{4}} + \cancel{\frac{3}{2}} - 1\right) = 2$

b) $\int x^2 e^{2x}\,dx \quad = \quad x^2\cdot\frac{e^{2x}}{2} - \int 2x\cdot\frac{e^{2x}}{2}\,dx = x^2\cdot\frac{e^{2x}}{2} - \int x\,e^{2x}\,dx$

$= x^2\frac{e^{2x}}{2} - \left[x\frac{e^{2x}}{2} - \int 1\cdot\frac{e^{2x}}{2}\,dx\right] = x^2\frac{e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + C$

Use Integration by parts twice

c) $\int \sin^9(x)\cos(x)\,dx \quad = \int u^9\,du = \frac{u^{10}}{10} + C = \frac{(\sin x)^{10}}{10} + C$

$u = \sin x$

$du = \cos x\,dx$

(Problem 1 continued)

d) $\displaystyle\int \frac{3x^2-2x+3}{(x^2-1)(x^2+1)}\,dx$ $\qquad \dfrac{3x^2-2x+3}{(x^2-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$

$\Rightarrow\ 3x^2-2x+3 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$

$x=-1 \Rightarrow 8=-4B \Rightarrow B=-2$

$x=1 \Rightarrow 4=4A \Rightarrow A=1$

$3x^2-2x+3 = 3-D + x(-1-C) + x^2(3+D) + x^3(C-1)$

$1:\ 3=3-D$

$x:\ -2=-1-C$

$x^2:\ 3=3+D$

$x^3:\ 0=C-1$

$\Rightarrow \begin{cases} C=1 \\ D=0 \end{cases} \Rightarrow \displaystyle\int \frac{3x^2-2x+3}{(x^2-1)(x^2+1)}\,dx = \int\frac{dx}{x-1} - \int\frac{2dx}{x+1} + \int\frac{x}{x^2+1}\,dx$

$= \ln|x-1| - 2\ln|x+1| + \frac{1}{2}\ln|x^2+1| + C$

$\displaystyle\int\frac{x}{x^2+1}\,dx = \frac{1}{2}\int\frac{2x}{x^2+1}\,dx = \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$

$u = x^2+1$
$du = 2x\,dx$

e) $\displaystyle\int_0^4 \frac{dx}{(x-2)^2}$ The function $\dfrac{1}{(x-2)^2}$ is discontinuous at $x=2$ so this is an improper integral.

$\displaystyle\int_0^4 \frac{dx}{(x-2)^2} = \int_0^2 \frac{dx}{(x-2)^2} + \int_2^4 \frac{dx}{(x-2)^2}$ (both integrals are improper).

$\displaystyle\int_0^2 \frac{dx}{(x-2)^2} = \lim_{t\to 2^-}\int_0^t \frac{dx}{(x-2)^2} = \lim_{t\to 2^-}\int_0^t (x-2)^{-2}\,dx$

$\displaystyle = \lim_{t\to 2^-} \left.\frac{(x-2)^{-1}}{-1}\right|_0^t = \lim_{t\to 2^-} \frac{1}{2-t} - \frac{1}{2} = \infty$

$\Rightarrow$ the integral diverges

$\Rightarrow \displaystyle\int_0^4 \frac{dx}{(x-2)^2}$ diverges as well.

Problem 2. (18 points) Evaluate the following expressions:

a) $\frac{d}{dx}\left(\int_3^{e^x} \arctan(\ln(t))\,dt\right) = \arctan(\ln(e^x))e^x - 0 = \arctan(x)e^x$

b) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}(1+\frac{k}{n})$

<u>Solution 1</u>: $\int_0^1 (1+x)dx = x + \frac{x^2}{2}\Big|_0^1 = 1 + \frac{1}{2} = \frac{3}{2}$

$$\boxed{\begin{array}{l} x = \frac{k}{n} \qquad \Delta x = \frac{1}{n} \\ k=1 \Rightarrow x = \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \\ k=n \Rightarrow x = \frac{n}{n} = 1 \\ x \in [0,1]. \end{array}}$$

<u>Solution 2</u>: $\int_1^2 x\,dx = \frac{x^2}{2}\Big|_1^2 = \frac{4-1}{2} = \frac{3}{2}$

$$\boxed{\begin{array}{l} x = 1 + \frac{k}{n} \qquad \Delta x = \frac{1}{n} \\ k=1 \Rightarrow x = 1 + \frac{1}{n} \xrightarrow[n\to\infty]{} 1 \\ k=n \Rightarrow x = 1 + \frac{n}{n} = 2 \\ \Rightarrow x \in [1,2] \end{array}}$$

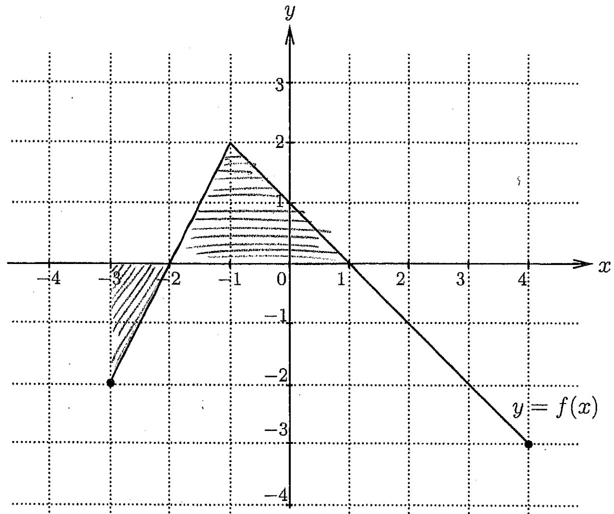
<u>Solution 3</u>: $\sum_{k=1}^{n}\frac{1}{n}\left(1+\frac{k}{n}\right) = \sum_{k=1}^{n}\frac{1}{n} + \sum_{k=1}^{n}\frac{k}{n^2}$

$$= \frac{1}{n}\sum_{k=1}^{n} 1 + \frac{1}{n^2}\sum_{k=1}^{n} k$$

$$= \frac{1}{n}\cdot n + \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = 1 + \frac{1}{2}\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(1+\frac{k}{n}\right) = \lim_{n\to\infty} 1 + \frac{1}{2}\left(1+\frac{1}{n}\right) = 1 + \frac{1}{2} = \frac{3}{2}$$

Problem 3. (10 points) Consider the function $f(x)$ graphed below:



Now define a **new** function $F(x) = \int_{-3}^{x} f(t)\,dt$ on the interval $[-3, 4]$.

For what value of x does F have a (global) maximum value? What is the maximum value?

Justify all your answers!

By the Fundamental Theorem of Calculus $F'(x) = \frac{d}{dx}\int_{-3}^{x} f(t)\,dt = f(x)$.

F is increasing when $F'(x) > 0$, that is when $f(x) > 0$
decreasing when $F'(x) < 0$, that is when $f(x) < 0$.

So F is increasing on $(-2, 1)$ and decreasing on $(-3,-2) \cup (1,4)$

The function F has a local maximum at $x = 1$ where $F'(x) = 0$.

$$F(1) = \int_{-3}^{1} f(x)\,dx = \underbrace{\int_{-3}^{-2} f(x)\,dx}_{= -\frac{1}{2}\cdot 1\cdot 2 = -1} + \underbrace{\int_{-2}^{1} f(x)\,dx}_{= \frac{1}{2}\cdot 2\cdot 3 = 3} = -1 + 3 = 2$$

To see if this is the global maximum, we compare $F(1)$ with the value of F at the endpoints.

$$F(-3) = \int_{-3}^{-3} f(x)\,dx = 0 \quad < F(1)$$

$$F(4) = \int_{-3}^{4} f(x)\,dx = \underbrace{\int_{-3}^{1} f(x)\,dx}_{= F(1) = 2} + \underbrace{\int_{1}^{4} f(x)\,dx}_{< 0 \text{ since } f \text{ is negative}} \quad < F(1)$$

$\Rightarrow$ The global maximum is attained at $\underline{x = 1}$.

Problem 4. (30 points) The region R in the first quadrant bounded by $y = \sin(x)$ and $y = \cos(x)$ on the interval $[0, \pi/4]$ is shown to the right.

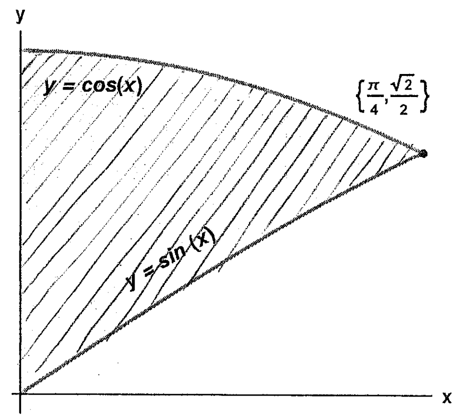
a) Find the area of the region R.

$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\, dx$$

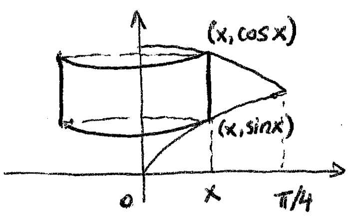
$$= \sin x + \cos x \Big|_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)$$

$$= \sqrt{2} - 1$$



b) Find the volume of the solid of revolution that results when R is revolved about the y-axis, using the Shell Method. Draw a typical cylindrical shell.



A typical cylindrical shell has radius $r = x$ and height $h = \cos x - \sin x$

Its area is $A(x) = 2\pi r h = 2\pi x(\cos x - \sin x)$

$$\text{Volume} = \int_0^{\pi/4} 2\pi x(\cos x - \sin x)\, dx = 2\pi \int_0^{\pi/4} x(\cos x - \sin x)\, dx$$

Use integration by parts $\quad u = x \quad dv = (\cos x - \sin x)dx$

$\quad du = dx \quad v = \sin x + \cos x$

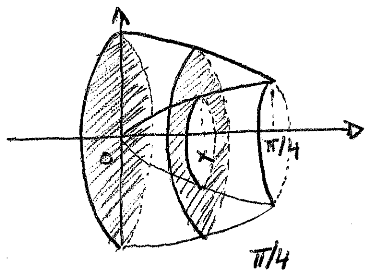
$$= 2\pi x(\sin x + \cos x)\Big|_0^{\pi/4} - 2\pi \int_0^{\pi/4} (\sin x + \cos x)\, dx$$

$$= 2\pi \cdot \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - 2\pi(-\cos x + \sin x)\Big|_0^{\pi/4}$$

$$= \frac{\pi^2\sqrt{2}}{2} - 2\pi\left(\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (-1+0)\right) = \frac{\pi^2\sqrt{2}}{2} - 2\pi$$

(*Problem 4 continued*)

c) Find the volume of the solid of revolution that results when R is revolved about the x-axis, using the Disk/Washer Method. Draw a typical washer.



A typical washer has outer radius $R = \cos x$
and inner radiur $r = \sin x$

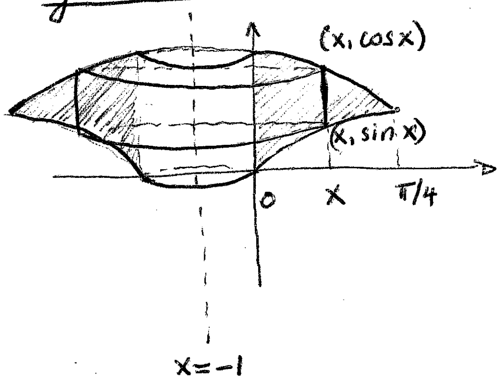
Its area is $A(x) = \pi(R^2 - r^2) = \pi(\cos^2 x - \sin^2 x)$

$$\text{Volume} = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\,dx = \pi \int_0^{\pi/4} \cos(2x)\,dx = \pi \frac{\sin(2x)}{2}\Big|_0^{\pi/4}$$

$$= \frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) = \frac{\pi}{2}$$

d) Set up (but do not integrate!) the integral that gives the volume when R is revolved about the vertical line $x = -1$.

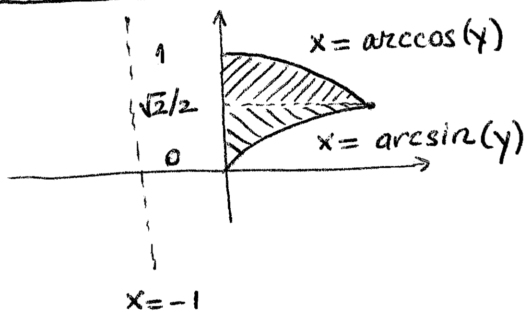
Cylindrical Shells:



A typical shell has radius $r = x - (-1) = x + 1$
height $h = \cos x - \sin x$
area $A = 2\pi(x+1)(\cos x - \sin x)$.

$$\text{Volume} = \int_0^{\pi/4} 2\pi (x+1)(\cos x - \sin x)\,dx$$

Disk/Washer Method



$$\text{Volume} = \int_0^{\sqrt{2}/2} \pi \, \operatorname{arsin}(y)^2\,dy + \int_{\sqrt{2}/2}^{1} \pi \arccos(y)^2\,dy$$

Problem 5. (12 points) Evaluate the integral $\int \frac{x^3}{\sqrt{x^2-4}}\,dx$. Simplify your final answer.

Trigonometric Substitution $\quad x = 2\sec\theta$

$$dx = 2\sec\theta\tan\theta\,d\theta$$

$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4\tan^2\theta} = 2\,|\tan\theta|$$

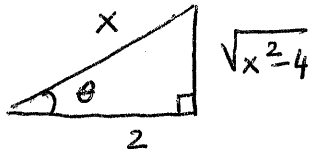
$$\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{8\sec^3\theta}{2\tan\theta}\cdot 2\sec\theta\tan\theta\,d\theta = 8\int \sec^4\theta\,d\theta$$

$$8\int \sec^4\theta\,d\theta = 8\int \sec^2\theta\sec^2\theta\,d\theta = 8\int (\tan^2\theta + 1)\sec^2\theta\,d\theta$$

$$u = \tan\theta$$

$$du = \sec^2\theta\,d\theta$$

$$= 8\int (u^2+1)\,du = 8\left(\frac{u^3}{3} + u\right) + C = 8\left(\frac{\tan^3\theta}{3} + \tan\theta\right) + C$$



Construct a right triangle with an angle θ with $\sec\theta = \frac{x}{2}$ (or equivalently with $\cos\theta = \frac{2}{x}$)

From the right triangle we evaluate $\tan\theta = \frac{\sqrt{x^2-4}}{2}$

Therefore $\int \frac{x^3}{\sqrt{x^2-4}}\,dx = 8\left(\frac{\sqrt{x^2-4}}{2}\right)^3\cdot\frac{1}{3} + 8\cdot\frac{\sqrt{x^2-4}}{2} + C$

$$= \frac{(x^2-4)\sqrt{x^2-4}}{3} + 4\sqrt{x^2-4} + C$$

Problem 6. (18 points)

a) Calculate the arc length of the curve $y = 2x^{3/2} + 1$ over the interval $[0, \frac{1}{3}]$.

$$\text{Arclength} = \int_0^{1/3} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{1/3} \sqrt{1+9x}\,dx = \frac{1}{9}\int_1^4 \sqrt{u}\,du$$

$$y = 2x^{3/2}+1$$

$$\frac{dy}{dx} = 2\cdot\frac{3}{2}\cdot x^{3/2-1} = 3\sqrt{x}$$

$$\boxed{\begin{array}{l} u = 1+9x \\ du = 9dx \\ x=0 \Rightarrow u=1 \\ x=\frac{1}{3} \Rightarrow u = 1+9\cdot\frac{1}{3} = 4 \end{array}}$$

$$= \frac{1}{9}\int_1^4 u^{\frac{1}{2}}du = \frac{1}{9}\left.\frac{u^{3/2}}{3/2}\right|_1^4 = 2\,\frac{\sqrt{4}^3-1}{27} = \frac{14}{27}$$

b) Find the average value of the function $y = \sin(x)e^{\cos(x)}$ over the interval $[\frac{\pi}{2}, \pi]$.

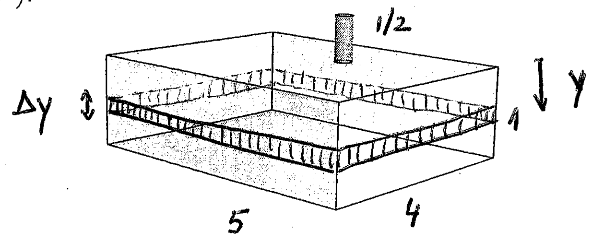
$$\text{Average} = \frac{1}{\pi - \frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi} \sin x\, e^{\cos x}\,dx$$

Substitution

$$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \\ x = \frac{\pi}{2} \Rightarrow u = \cos\frac{\pi}{2} = 0 \\ x = \pi \Rightarrow u = \cos\pi = -1 \end{array}}$$

$$= \frac{2}{\pi}\int_0^{-1} -e^{+u}du = \frac{2}{\pi}\,e^u\Big|_{-1}^{0} = \frac{2}{\pi}\left(e^0 - e^{-1}\right) = 2\cdot\frac{1-\frac{1}{e}}{\pi}$$

Problem 7. (14 points) A rectangular tank 5m long, 4m wide, and 1m deep is full of water. Find the work needed to pump the water out of the tank through a small spout at the top, with height 1/2 m. (The density of water is $\rho = 1000 kg/m^3$ and the constant of gravitational acceleration is $g = 9.8 m/s^2$).



Let the positive y-axis point downward from the top of the tank.

The volume of one layer of water of height Δy is $V(y) = 4 \cdot 5 \Delta y = 20 \Delta y \ (m^3)$

The force needed to lift the layer is

$$F = m \cdot g = \rho \cdot V \cdot g = 20 \rho g \Delta y \quad (N)$$

Each layer of water must be lifted $y + \frac{1}{2}$ meters, so the work needed is

$$W = F(y + \tfrac{1}{2}) = 20 \rho g (y + \tfrac{1}{2}) \Delta y \quad (J)$$

The total work needed to empty the tank is

$$W = \int_0^1 20 \rho g \left(y + \tfrac{1}{2}\right) dy = 20 \rho g \int_0^1 \left(y + \tfrac{1}{2}\right) dy$$

$$= 20 \rho g \left(\frac{y^2}{2} + \frac{y}{2} \right) \Big|_0^1 = 20 \rho g$$

$$= 20 \cdot 1000 \cdot 9.8 = 19.6 \cdot 10^5 \simeq 2 \cdot 10^6 \quad (J)$$

Problem 8. (10 points) Determine whether the following statements are true or false. Circle your response and give a brief explanation (a reason why it's true or an example where it fails).

a) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$ If f is a continuous function on $[a, b]$ such that $\int_a^b f(x)\,dx = 0$, then there exists at least one point x in (a, b) for which $f(x) = 0$.

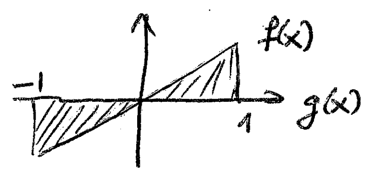
True: By the Mean Value Theorem, there exists a point c in (a,b) for which $f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx = \frac{1}{b-a} \cdot 0 = 0$

b) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$ Let f and g be two integrable functions on $[a, b]$. The definite integral $\int_a^b (f(x) - g(x))\,dx$ represents the area of the region between the graphs of f and g.

True if $f(x) \geq g(x)$ for all x in $[a,b]$.

False in general

Counterexample: $f(x) = x$ on $[-1, 1]$.
$g(x) = 0$



$$\text{Area} = \int_{-1}^{1} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx = \frac{1}{2} + \frac{1}{2} = 1$$

Whereas $\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

So in this case $\text{Area} \neq \int_{-1}^{1} (f(x) - g(x))\,dx$