Name: SOLUTIONS Id: FALL '16

10 pts 1. Calcuate the integral $\int_{-1}^{1} x^3 + 3x^2 - \frac{1}{1+x^2}\,dx$.

If the integral is improper and it does not converge, write "Diverges" (and justify).

$$= \frac{1}{4}x^4 + x^3 - \arctan(x)\Big|_{-1}^{1}$$

$$= \left(\frac{1}{4} + 1 - \pi/4\right) - \left(\frac{1}{4} - 1 + \frac{\pi}{4}\right)$$

$$= \boxed{2 - \pi/2}$$

10 pts 2. Calcuate the indefinite integral $\int we^{2w}\,dw$. $= \frac{1}{2}we^{2w} - \frac{1}{2}\int e^{2w}\,dw$

BY PARTS: $u = w \quad dv = e^{2w}dw$

$du = dw \quad v = \frac{1}{2}e^{2w}$

$$= \boxed{\frac{1}{2}we^{2w} - \frac{1}{4}e^{2w} + C}$$

10 pts 3. Calcuate the integral $\int_{1}^{\infty} \frac{du}{(2u-1)^2}$.

If the integral is improper and it does not converge, write "Diverges" (and justify).

LET $w = 2u - 1$ WHEN $u = 1$, $w = 1$

$dw = 2du$ WHEN $u = \infty$, $w = \infty$

$$= \frac{1}{2}\int_{1}^{\infty}\frac{dw}{w^2} = \frac{-1}{2w}\Big|_{1}^{\infty} = \lim_{w\to\infty}\frac{-1}{2w} + \frac{1}{2}$$

$$= 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

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15 pts 4. Calculate the indefinite integral $\int \frac{3dx}{(2x+1)(x-1)}$. $= \int\left(\frac{A}{2x+1} + \frac{B}{x-1}\right)dx$

$3 = A(x-1) + B(2x+1)$

IF $x=1$, $3 = 0 + 3B$, SO $B = 1$.

IF $x=-\frac{1}{2}$, $3 = A\cdot\frac{-3}{2} + 0$, SO $A=-2$

[ALTERNATIVELY, WE HAVE

$3 = Ax - A + 2Bx + B$, SO $A+2B=0$
$-A+B=3$

ADDING GIVES $3B=3$, SO $A=-2$
$B=1$ ]

$= \int \frac{-2}{2x+1} + \frac{3}{x-1}\,dx = \boxed{-\ln|2x+1| + \ln|x-1| + C}$

15 pts 5. Calculate the indefinite integral $\int \sin^4(2t)\cos^3(2t)\,dt$

REWRITE USING $\cos^2(2t) = -\sin^2(2t)+1$

SO $\int \sin^4(2t)(1-\sin^2(2t))\cos(2t)\,dt$

$= \int \sin^4(2t)\cos(2t) - \sin^6(2t)\cos(2t)\,dt$

$\boxed{\text{LET } u = \sin 2t, \text{ so } du = 2\cos 2t}$

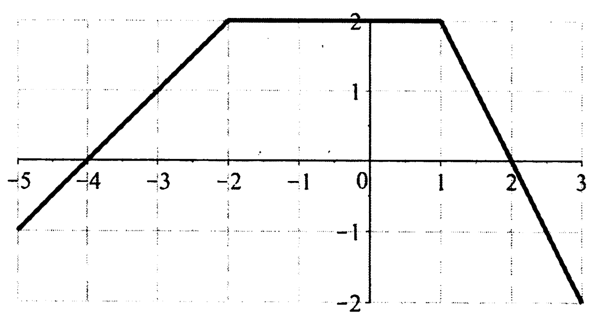
$= 2\int u^4 - u^6\,du = \frac{1}{2}\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + c$

$= \boxed{\frac{1}{2}\left(\frac{\sin^5(2t)}{5} - \frac{\sin^7(2t)}{7}\right) + c}$

6. The function $f(x)$ has the graph shown at right, for $-5 \le x \le 3$.

Define another function $g(x)$ by

$$g(x) = \int_{-5}^{x} f(t)\,dt.$$



5 pts (a) For what $x$ does $g(x)$ take on its maximum value for $-5 \le x \le 3$?
(If there is no maximum, write "None"; if there are several such $x$, list them all.)

SINCE CRITICAL POINTS ARE WHERE $f(t)=0$, CHOICES ARE $-5$, $-4$, $2$, AND $3$.

2 IS A LOCAL MAXIMUM, AND ALSO THE GLOBAL ONE.

5 pts (b) What is $g(2)$? (If it is not defined, write "DNE".)

COUNTING BOXES, WE GET

$$g(2) = \boxed{9.5} = \frac{19}{2}.$$

5 pts (c) What is the minimum value of $g(x)$ for $-5 \le x \le 3$?
(If there is no minimum, write "None".)

$g(-5) = -\frac{1}{2}$ IS THE SMALLEST $g(x)$ TAKES ON IN THE RANGE.

5 pts (d) What is the largest interval (for $x$ between $-5$ and 3) on which $g(x)$ is concave down?
(If there is no such interval, write "None".)

NOTE THAT $g''(x) = f'(x)$,

SO $g''(x)$ IS POSITIVE FOR $-5<x<-2$
ZERO FOR $-2<x<1$
AND NEGATIVE FOR $1<x<3$.

ANSWER $\boxed{1 < x < 3}$

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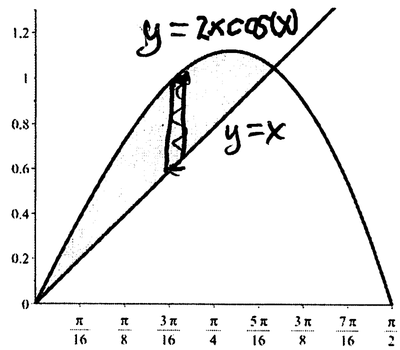
[20 pts] 7. Consider the region lying above the graph of $y = x$, below the graph of $y = 2x\cos(x)$, to the right of $x = 0$, and to the left of $x = \frac{\pi}{2}$. See the figure at right (note that the region does not contain all $x$ values between 0 and $\frac{\pi}{2}$).




**Find the area of this region.**

WHERE DO THEY CROSS?

$2x\cos x = x$

IF $x=0$ OR $\cos x = \frac{1}{2}$

$\Rightarrow \quad x = \pi/3$

AREA IS $\int_0^{\pi/3} 2x\cos x - x)\,dx$

$u = x \quad v = \sin(x)$

$du = dx \quad dv = \cos(x) \qquad -\left(\frac{x^2}{2}\right)\Big|_0^{\pi/3}$

$\left(x\sin x - \int^{\pi/3} \sin(x)\,dx\right) - \frac{\left(\frac{\pi}{3}\right)^2}{2} - 0$

$(x\sin x + \cos x)\Big|_0^{\pi/3} - \frac{\pi^2}{18}$

$\left(\frac{\pi}{3}\cdot\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\right) - (0 + \cos 0) - \frac{\pi^2}{18}$

$\frac{\pi}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2} - 1 - \frac{\pi^2}{18}$

$\frac{\pi\sqrt{3}+3}{6} - 1 - \frac{\pi^2}{18}$

$\frac{\pi\sqrt{3}-3}{6} - \frac{\pi^2}{18}$

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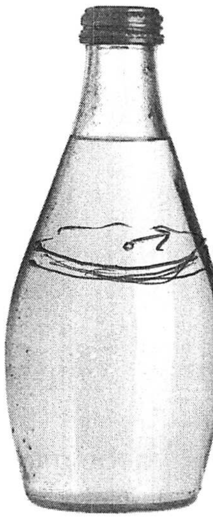
Do **any four** of the five questions 8-12. Cross out the one you don't want graded.

| 20 pts |

8. A bottle 12 inches high has its radius $r$ (in inches) at height $h$ recorded in the table at right. **Use Simpson's rule to estimate the volume of the bottle.**

(If you don't remember Simpson's rule, you may use the Trapezoid rule instead, but you will get a maximum of 15 points. Or, use left or right endpoints for half credit. **Clearly indicate which method you are using!**).

| $h$ | 0 | 3 | 6 | 9 | 12 |
|---|---|---|---|---|---|
| $r$ | 2 | 3 | 3 | 2 | 1 |



Hint: First write an integral representing the volume in terms of some function $r(h)$, then use Simpson's rule to approximate it.

SLICE HORIZONTALLY.

A TYPICAL CROSS SECTION AT $h$ IS A DISK OF RADIUS $r(h)$.

AREA IS $\pi(r(h))^2$

SO VOLUME IS

$$\int_0^{12} \pi (r(h))^2 \, dh.$$

$h_0 = 0$, $h_1 = 3$, $h_2 = 6$, $h_3 = 9$, $h_4 = 12$

WE KNOW $r(h)$ AT $0, 3, 6, 9, 12$

SO WE TAKE $n = 4$ FOR SIMPSONS. $b - a = 12$.

THEN WE GET

$$\frac{12}{4\cdot 3}(\pi)\left(1\cdot 2^2 + 4\cdot 3^2 + 2\cdot 3^2 + 4\cdot 2^2 + 1\cdot 1^2\right)$$

$$= \pi(4 + 36 + 18 + 16 + 1)$$

$$= \boxed{75\pi \text{ in}^3.}$$

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Do **any four** of the five questions 8-12. Cross out the one you don't want graded.

10 pts 9. (a) Write an integral that represents the arc length of the graph of $y = 2x + \frac{x^3}{3}$ for $0 \le x \le 3$. **You do not need to calculate the value of the integral!**

RECALL THAT THE ARC LENGTH OF $y=f(x)$ FROM a TO b

IS $\int_a^b \sqrt{1+(f'(x))^2}\,dx$

SO:

$$\boxed{AL = \int_0^3 \sqrt{1+(2+x^2)^2}\,dx}$$

10 pts (b) Find the average value of the function $f(x) = x^2 \sin(2x)$ over the domain $0 \le x \le \frac{\pi}{2}$.

AVERAGE VALUE IS $\frac{1}{b-a}\int_a^b x^2 \sin(2x)\,dx$

$$\frac{2}{\pi}\int_0^{\pi/2} x^2 \sin(2x)\,dx \qquad \begin{array}{ll} u = x^2 & dv = \sin(2x)\,dx \\ du = 2x\,dx & v = -\frac{1}{2}\cos(2x) \end{array}$$

$$\begin{array}{ll} u = x & dv = \cos 2x\,dx \\ du = dx & v = \frac{1}{2}\sin 2x \end{array}$$

$$= \frac{2}{\pi}\left(-\frac{1}{2}x^2 \cos 2x \Big|_0^{\pi/2} + \int_0^{\pi/2} x \cos 2x\,dx\right)$$

$$= \frac{2}{\pi}\left(-\frac{1}{2}x^2\cos(2x)\Big|_0^{\pi/2} + \frac{1}{2}x\sin(2x)\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} \sin 2x\,dx\right)$$

$$= \frac{2}{\pi}\left(-\frac{1}{2}\cdot\frac{\pi^2}{4}(-1) + 0 - \frac{1}{4}\cos(2x)\Big|_0^{\pi/2}\right) = \boxed{\frac{\pi}{4} - \frac{1}{\pi}}$$

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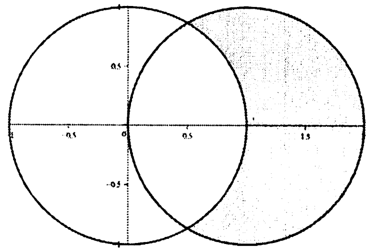
Do **any four** of the five questions 8-12. Cross out the one you don't want graded.

20 pts 11. Find the area of the region that lies inside the circle of radius one centered at $(1, 0)$, but outside the circle of radius one centered at the origin.

You can do this either in polar coordinates (where the two curves are given by $r = 2\cos\theta$ and $r = 1$) or in rectangular coordinates (where the curves are given by $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$). In the rectangular case, you need to cut up the area appropriately.



IN POLAR COORDS: CURVES CROSS WHEN $2\cos\theta = 1$

IE $\cos\theta = 1/2$

IE $\theta = \pm\pi/3$

AREA IS $\frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 - 1^2 \, d\theta$

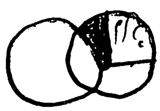
$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos^2\theta - 1 \, d\theta$

$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2(1+\cos 2\theta) - 1 \, d\theta$

$= \frac{1}{2}(\theta + \sin 2\theta)\Big|_{-\pi/3}^{\pi/3} = \frac{\pi}{3} + \sin(2\pi/3)$

$= \boxed{\pi/3 - \frac{\sqrt{3}}{2}}$

IN RECT. COORDS,



SPLIT IT INTO PART WITH $x<1$ (BLACK) AND $x>1$ (WHITE) THEN DOUBLE IT.

FOR $x<1$,

UPPER CURVE IS $y = \sqrt{1-(x-1)^2}$, LOWER IS $y = \sqrt{1-x^2}$.

THEY CROSS AT $x = 1/2$.

GET $\underbrace{\int_{1/2}^{1} \sqrt{1-(x-1)^2} - \sqrt{1-x^2}\, dx}_{= \sqrt{3}/4 - \pi/12}$

OTHER PIECE IS A $\frac{1}{4}$ CIRCLE,

$A = \pi/4$.



SO ADD THEM TOGETHER, DOUBLE IT, YOU GET SAME ANSWER

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Do **any four** of the five questions 8-12. Cross out the one you don't want graded.

12. Wind speed can be modeled by the Rayleigh distribution $f(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ where $\sigma$ is the most common wind speed (the mode).

10 pts (a) If the most common wind speed is a breeze of 5 knots, what is the probability that the wind speed will be more than 20 knots? Do not approximate $e$, logs, or square roots.

$\sigma = 5$, AND P(SPEED > 20) IS GIVEN BY

$$\int_{20}^{\infty} \frac{x}{25} e^{-x^2/50} \, dx$$

LET $u = +x^2/50$
SO $du = +x/25 \, dx$

$x = 20 \Rightarrow u = \frac{400}{50} = 8$
$x = \infty \Rightarrow u = \infty$

$$= \int_8^{\infty} +e^{-u} \, du$$

$$= -e^{-u}\Big|_8^{\infty} = \lim_{b \to \infty} \left(-e^{-b} + e^{-8}\right) = 0 + e^{-8} \; \boxed{= \frac{1}{e^8}}$$

10 pts (b) Again assuming $\sigma = 5$, calculate the median[1] wind speed. Do not give an approximation.

MUST SOLVE

$$\int_0^{m} \frac{x}{25} e^{-x^2/50} \, dx = \frac{1}{2}$$

$$= \int_0^{m^2/50} e^{-u} \, du$$

$$= -e^{-m^2/50} + 1$$

SO $e^{-m^2/50} = 1/2$

TAKE ln OF BOTH SIDES

$-m^2/50 = -\ln 2$

$m^2 = 50 \ln 2$

$$\boxed{m = \sqrt{50 \ln 2}}$$

[1] Recall that the **median** of a probability density function $f(x)$ is the number $m$ so that the probability that $x \geq m$ is $\frac{1}{2}$ (and so also the probability that $x < m$ is $\frac{1}{2}$).