Math 126 Solutions to Midterm 2 (instrumental)

1. Determine these EASY antiderivatives. You should be able to do these **very well**. In these problems, no justification is needed. Remember the '+C'.

6 pts (a) $\displaystyle\int \frac{4}{x}\,dx$

Solution:

$$\int \frac{4}{x}\,dx = \boxed{4\ln|x| + C}$$

6 pts (b) $\displaystyle\int 8\sin(x)\,dx$

Solution:

$$\int 8\sin(x)\,dx = \boxed{-8\cos(x) + C}$$

6 pts (c) $\displaystyle\int e^{2x}\,dx$

Solution:

$$\int e^{2x}\,dx = \boxed{\frac{1}{2}e^{2x} + C}$$

6 pts (d) $\displaystyle\int \frac{4}{t^2+1}\,dt$

Solution:

$$\int \frac{4}{t^2+1}\,dt = \boxed{4\arctan(t) + C}$$

6 pts (e) $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du$

Solution:

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \boxed{\arcsin(u) + C}$$

2. In this question we tell you which method we suggest you use. Use the back of the previous page if you need more space.

15 pts (a) Suggested method: substitution $\displaystyle\int \frac{x}{1+x^2}\,dx$

Solution: Make the substitution $u = 1 + x^2$, so that $du = 2dx$, or $\frac{1}{2}du = dx$. Then

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|1+x^2| + C}$$

Note that since $1 + x^2 > 0$ for all x, the absolute value is not necessary; the answer $\dfrac{\ln(1+x^2)}{2} + C$ is fine, too.

15 pts (b) Suggested method: substitution $\displaystyle\int \frac{e^{\sqrt{z+3}}}{\sqrt{z+3}}\,dz$

Solution: Make the substitution $u = \sqrt{z+3}$. Then

$$du = \frac{1}{2\sqrt{z+3}}\,dz \qquad \text{or} \qquad 2du = \frac{dz}{\sqrt{z+3}}.$$

Thus,

$$\int \frac{e^{\sqrt{z+3}}}{\sqrt{z+3}}\,dz = 2\int e^u\,du = 2e^u + C = \boxed{2e^{\sqrt{z+3}} + C}$$

15 pts (c) Suggested method: substitution $\displaystyle\int \frac{\ln(y)}{y}dy$

Solution: Here, we let $u = \ln(y)$ and so $du = \dfrac{dy}{y}$. This means we have

$$\int \frac{\ln(y)}{y}dy = \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{\big(\ln(y)\big)^2}{2} + C}$$

3. In this question we tell you which method we suggest you use. Use the back of the previous page if you need more space.

15 pts (a) Suggested method: integration by parts $\int x^4 \ln(x)\, dx$

Solution: Take $u = \ln(x)$ and $dv = x^4\, dx$. Then $du = \frac{1}{x}\, dx$ and $v = \frac{x^5}{5}$. So:

$$\int x^4 \ln(x)\, dx = \frac{x^5 \ln(x)}{5} - \frac{1}{5}\int x^5 \cdot \frac{1}{x}\, dx = \frac{x^5 \ln(x)}{5} - \frac{1}{5}\int x^4\, dx = \boxed{\frac{x^5 \ln(x)}{5} - \frac{x^5}{25} + C}$$

15 pts (b) Suggested method: integration by parts $\int xe^{3x}\, dx$

Solution: Take $u = x$ and $dv = e^{3x}\, dx$. Then $du = dx$ and $v = \frac{1}{3}e^{3x}$, and so we have

$$\int xe^{3x}\, dx = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}\, dx = \boxed{\frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C}$$

15 pts (c) Suggested method: integration by parts $\int \sin(x)e^{2x}\, dx$

Solution: Take $u = e^{2x}$ and $dv = \sin(x)\, dx$. Then $du = 2e^{2x}\, dx$ and $v = -\cos(x)$. So we have

$$\int \sin(x)e^{2x}\, dx = -\cos(x)e^{2x} + 2\int \cos(x)e^{2x}\, dx$$

(we have a $+$ before the integral because we were subtracting a negative). To do the second integral, we take $u = e^{2x}$ and $dv = \cos x\, dx$. Then $du = 2e^{2x}\, dx$ and $v = \sin x$. This gives us

$$\int \sin(x)e^{2x}\, dx = -\cos(x)e^{2x} + 2\Big(\sin(x)e^{2x} - 2\int \sin(x)e^{2x}\, dx\Big)$$

Multiplying out gives

$$\int \sin(x)e^{2x}\, dx = -\cos(x)e^{2x} + 2\sin(x)e^{2x} - 4\int \sin(x)e^{2x}\, dx$$

or, equivalently,

$$5\int \sin(x)e^{2x}\, dx = -\cos(x)e^{2x} + 2\sin(x)e^{2x} + C$$

Thus, we have

$$\int \sin(x)e^{2x}\, dx = \boxed{\frac{-\cos(x)e^{2x} + 2\sin(x)e^{2x}}{5} + C}$$

4. Determine the following antiderivatives. Use the back of the previous page if you need more space.

15 pts (a) $\int \sin^3(x)\,dx$

Solution: We use the identity $\sin^2(x) = 1 - \cos^2(x)$ to get

$$\int \sin^3(x)\,dx = \int \left(1 - \cos^2(x)\right)\sin(x)\,dx.$$

Now take $u = \cos(x)$ and $du = -\sin(x)\,dx$, giving

$$\int \sin^3(x)\,dx = -\int (1-u^2)\,du = -u + \frac{u^3}{3} + C = \boxed{\frac{\cos^3(x)}{3} - \cos(x) + C}$$

15 pts (b) $\int \frac{1}{\sec(6x)}\,dx$

Solution:

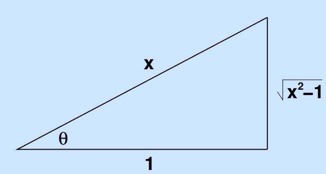
$$\int \frac{1}{\sec(6x)}\,dx = \int \cos(6x)\,dx = \boxed{\frac{1}{6}\sin(6x) + C}$$

15 pts (c) $\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$

Solution: Take $x = \sec\theta$ so $dx = \sec\theta\tan\theta\,d\theta$. Then we have

$$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\sec^2\theta - 1}} = \int \frac{\tan\theta\,d\theta}{\sec\theta\sqrt{\tan^2\theta}} = \int \frac{d\theta}{\sec\theta} = \int \cos\theta d\theta$$

This means we have $\sin\theta + C$ as our answer, but of course we need the answer in terms of x. Recall that we took $x = \sec\theta$, and so $\sin\theta = \frac{\sqrt{x^2-1}}{x}$ (see figure). Thus, we have shown



$$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \boxed{\frac{\sqrt{x^2-1}}{x} + C}$$

5. Evaluate these definite integrals. Use the back of the previous page if you need more space.

15 pts (a) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\,dx$

Solution: We use partial fractions:

$$\frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$$

so $\quad 1 = A(1-x) + B(1+x).$ Thus

$$A+B=1 \qquad -A+B=0 \qquad \text{hence} \quad A=\frac{1}{2},\, B=\frac{1}{2}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1/2}{1+x} + \frac{1/2}{1-x}\,dx = \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x|\Big|_{-1/2}^{1/2}$$

$$= \frac{1}{2}\left[\ln\left(\frac{3}{2}\right) - \ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right)\right]$$

$$= \ln\left(\frac{3}{2}\right) - \ln\left(\frac{1}{2}\right) = \boxed{\ln(3)}$$

15 pts (b) $\displaystyle\int_{-100}^{100} \frac{\sin^{21}(x)}{1+e^{x^2}}dx$

Solution: Since $\dfrac{\sin^{21}(x)}{1+e^{x^2}}$ is an odd function and the bounds are symmetric with respect to 0, the value of the integral is $\boxed{0}$.

15 pts (c) $\displaystyle\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$

Solution: Let $u = 4-x^2$ so that $du = -2x\,dx$. When $x=0$, $u=4$ and when $x=1$, $u=3$. Thus we have

$$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx = -\int_4^3 \frac{du}{2\sqrt{u}} = -\sqrt{u}\Big|_4^3 = -\sqrt{3}+\sqrt{4} = \boxed{2-\sqrt{3}}.$$

6. Since $\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(1) = \frac{\pi}{4}$, evaluating the integral $\int_0^1 \frac{4}{1+x^2}\,dx$ gives π.

20 pts

(a) Use Simpson's rule with 2 intervals to estimate $\int_0^1 \frac{4}{1+x^2}\,dx$.

Solution: Since there are two intervals, the width of each is $1/2$. Thus, Simpson's rule gives:

$$\frac{1}{3}\cdot\frac{1}{2}\Big(f(0)+4f(1/2)+f(1)\Big) = \frac{1}{6}\left(4+4\left(\frac{4}{1+1/4}\right)+2\right) = \boxed{\frac{94}{30}} \approx 3.13333$$

20 pts

(b) How many intervals are needed to estimate $\int_0^1 \frac{4}{1+x^2}\,dx = \pi$ within $.0001$ using the trapezoid rule?[1]

Solution: We use the information in the footnote. We need to determine n so that

$$\frac{1}{12n^2}K \leq .0001$$

where K is the maximum of the absolute value of the second derivative of $4/(1+x^2)$ for x between 0 and 1. Since $\left|\frac{4(6x^2-2)}{(1+x^2)^3}\right|$ is a decreasing function on this interval, the maximum occurs at $x=0$, so we take $K = |-8/1| = 8$.

To solve $\frac{8}{12n^2} \leq .0001$, we multiply both sides by $10000n^2$ to get

$$\frac{80000}{12} \leq n^2,$$

so n is the smallest integer bigger than $\sqrt{20000/3} \approx 81.6$.

Thus, $\boxed{n = 82}$.

[1]Use the following estimate for E_T using n intervals: If $|f''(x)| \leq K$ then $E_T \leq K\,\frac{(b-a)^3}{12n^2}$.

If $f(x) = \frac{1}{1+x^2}$, then $f''(x) = \frac{6x^2-2}{(1+x^2)^3}$