Math 126 Solutions to Midterm 1 (Atropos)

1. Evaluate each of the definite integrals below.

$\boxed{\text{5 pts}}$ (a) $\displaystyle\int_{-1}^{2} 3x^2 - 4x + 1\, dx$

Solution:

$$\int_{-1}^{2} 3x^2 - 4x + 1\, dx = x^3 - 2x^2 + x\Big|_{-1}^{2} = (8 - 8 + 2) - (-1 - 2 - 1) = 2 + 4 = 6.$$

$\boxed{\text{5 pts}}$ (b) $\displaystyle\int_{0}^{2} \left|1 - x^2\right|\, dx$

Solution: Note that $1 - x^2 > 0$ for $0 < x < 1$ and $1 - x^2 < 0$ for $x > 1$. This means that $x > 1$, $|1 - x^2| = -(1 - x^2) = x^2 - 1$, but if $x \le 1$, $|1 - x^2| = 1 - x^2$, and we split the integral into two pieces.

$$\begin{aligned}
\int_{0}^{2} \left|1 - x^2\right|\, dx &= \int_{0}^{1} (1 - x^2)\, dx + \int_{1}^{2} (x^2 - 1)\, dx \\
&= \left(x - \tfrac{1}{3}x^3\right)\Big|_{0}^{1} \quad + \left(\tfrac{1}{3}x^3 - x\right)\Big|_{1}^{2} \\
&= \Big((1 - \tfrac{1}{3}) - 0\Big) + \Big((\tfrac{8}{3} - 2) - (\tfrac{1}{3} - 1)\Big) = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{6}{3} = 2
\end{aligned}$$

$\boxed{\text{5 pts}}$ (c) $\displaystyle\int_{0}^{5} \sin(\pi x/5)\, dx$

Solution: Make the substitution $u = \frac{\pi x}{5}$. Then $du = \frac{\pi}{5}\, dx$, or $\frac{5}{\pi}\, du = dx$. When $x = 0$, $u = 0$, and when $x = 5$, $u = \pi$. Thus

$$\begin{aligned}
\int_{0}^{5} \sin(\pi x/5)\, dx &= \frac{5}{\pi} \int_{0}^{\pi} \sin(u)\, du \\
&= -\frac{5}{\pi} \cos(u)\Big|_{0}^{\pi} = -\frac{5}{\pi} \left(\cos(\pi) - \cos(0)\right) = -\frac{5}{\pi}(-1 - 1) = \frac{10}{\pi}
\end{aligned}$$

15 pts

2. Consider the function $f(x) = 1 + x^3$.

(a) Approximate the area lying under graph of $f(x)$, above the x-axis, and between the vertical lines $x = -1$ and $x = 5$ using 3 rectangles of equal width, evaluated at the **right endpoint** of each interval.

Solution: Since we are using 3 rectangles of equal width and we need to go from -1 to 5, each rectangle is of width 2. Thus, we have

$$2\Big(f(1) + f(3) + f(5)\Big) = 2\Big(2 + (28) + (126)\Big) = 2 \cdot 156 = 312$$

(b) Write a formula for the Riemann sum (using n rectangles of equal width) which represents the area under the graph of $f(x)$ for $-1 \leq x \leq 5$.

Solution: The width of each rectangle is $\frac{6}{n}$, and we start at $a = -1$, so we have $x_i = -1 + \frac{6i}{n}$. Thus, the appropriate Riemann sum (using right endpoints) is

$$\sum_{i=1}^{n} \frac{6}{n} \left(1 + \left(-1 + \frac{6i}{n}\right)^3\right)$$

(c) Compute the limit of the above Riemann sum as $n \to \infty$. You can do this directly or by calculating an integral.

The formulæ $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ or $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ could be helpful. Or not.

Solution: Doing it directly is not a good way to do this. It is much easier is to do the integral

$$\int_{-1}^{5} 1 + x^3 \, dx = x + \frac{x^4}{4}\Bigg|_{-1}^{5} = (5 + \tfrac{625}{4}) - (-1 + \tfrac{1}{4}) = 6 + \tfrac{624}{4} = 162.$$

If you insist on doing the direct computation, here it is. It isn't pretty.

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \left(1 + \left(-1 + \frac{6i}{n}\right)^3\right) &= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left(1 + \left(-1 + \frac{18i}{n} - \frac{108i^2}{n^2} + \frac{216i^3}{n^3}\right)\right) \\
&= \lim_{n\to\infty} \frac{6}{n} \left(\frac{18}{n} \sum_{i=1}^{n} i - \frac{108}{n^2} \sum_{i=1}^{n} i^2 + \frac{216}{n^3} \sum_{i=1}^{n} i^3\right) \\
&=6 \cdot \lim_{n\to\infty} \left(\frac{18(n^2+n)}{2n^2} - \frac{108(2n^3+3n^2+n)}{6n^3} + \frac{216(n^4+2n^3+n^2)}{4n^4}\right) \\
&=6 \cdot \lim_{n\to\infty} \left(\frac{9(n^2+n)}{n^2} - \frac{18(2n^3+3n^2+n)}{n^3} + \frac{54(n^4+2n^3+n^2)}{n^4}\right) \\
&=6 \cdot (9 - 36 + 54) = 6 \cdot 27 = 162.
\end{aligned}$$

There was a typo in the formula for $\sum k^2$ on the exam. You should get full credit if you used the wrong formula correctly. In that case, the answer would come out as 270.

3. Evaluate each of the indefinite integrals below.

5 pts (a) $\displaystyle\int \frac{5+x}{1+4x^2}\,dx$

Solution:

$$\int \frac{5+x}{1+4x^2}\,dx = \int \frac{5\,dx}{1+(2x)^2} + \int \frac{x\,dx}{1+4x^2} = \tfrac{5}{2}\arctan(2x) + \int \frac{1/8\,du}{u}$$

where we made the substitution $u = 1+4x^2$ (so $du = 8x\,dx$) in the second integral. Thus, the integral becomes

$$\tfrac{5}{2}\arctan(2x) + \frac{\ln(1+4x^2)}{8} + C$$

5 pts (b) $\displaystyle\int te^{-4t}\,dt$

Solution: Here, we integrate by parts, taking $u = t$ and $dv = e^{-4t}\,dt$. Thus $du = dt$ and $v = -\frac{1}{4}e^{-4t}$. So

$$\int te^{-4t}\,dt \quad = \quad -\frac{te^{-4t}}{4} + \frac{1}{4}\int e^{-4t}\,dt \quad = \quad -\frac{te^{-4t}}{4} - \frac{e^{-4t}}{16} + C$$

5 pts (c) $\displaystyle\int \tan w\,dw$

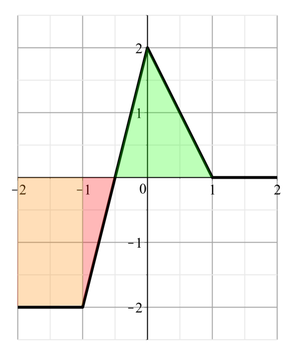
Solution:

$$\int \tan w\,dw = \int \frac{\sin w\,dw}{\cos w} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos w| + C$$

where we made the substution $u = \cos w$ so $du = -\sin w\,dw$.

4. Let $f(x)$ be the function whose graph is shown at right, and let

$$G(x) = \int_{-1}^{x} f(t)\,dt$$



6 pts (a) Calculate $G(-2)$, $G(-1)$, and $G(2)$.
If G is not defined, write something like "$G(5)$ DNE".

Solution: For all of these, we can just count the squares, remembering that those below the axis are negative.

$$\begin{aligned} G(-2) &= \int_{-1}^{-2} f(t)\,dt = -\int_{-2}^{-1} f(t)\,dt = -(-2) = 2 \\ G(-1) &= \int_{-1}^{-1} f(t)\,dt = 0 \\ G(2) &= \int_{-1}^{2} f(t)\,dt = \int_{-1}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt + \int_{1}^{2} f(t)\,dt = 0 + 1 + 0 = 1. \end{aligned}$$

5 pts (b) For what x with $-2 \le x \le 2$ is $G(x)$ increasing? (If there are none, write "none".)

Solution: Since $G'(x) = f(x)$, $G(x)$ decreases when $f(x) < 0$ and increases when $f(x) > 0$. Thus $G(x)$ is increasing for $-\frac{1}{2} < x < 1$.

5 pts (c) For what x with $-2 \le x \le 2$ is $G(x)$ concave up? (If there are none, write "none".)

Solution: Since $G''(x) = f'(x)$, $G(x)$ will be concave up when $f(x)$ has positive slope. That is, when $-1 < x < 0$.

10 pts 5. (a) Let $H(x) = \displaystyle\int_{x^2}^{1+x^3} \frac{1-t}{1+t}\,dt$. Calculate $H'(x)$.

Solution: Let $g(x) = \int_0^x \frac{1-t}{1+t}\,dt$, so $H(x) = g(1+x^3) - g(x^2)$. By the FTC, $g'(x) = \frac{1-x}{1+x}$. Using the chain rule, we have

$$H'(x) = g'(1+x^3)\cdot 3x^2 - g'(x^2)\cdot 2x = \frac{-x^3\cdot 3x^2}{2+x^3} - \frac{1-x^2}{1+x^2}\cdot 2x = -\frac{3x^5}{2+x^3} - \frac{2x-2x^3}{1+x^2}$$

(b) Find a function $g(x)$ so that $g'(x) = x\cos(x^2) + e^x$ and $g(0) = 4$.

Solution: We just want an antiderivative of $x\cos(x^2) + e^x$ with the proper constant. So

$$g(x) = \int (x\cos(x^2) + e^x)\,dx = e^x + \int x\cos(x^2)\,dx = e^x + \frac{1}{2}\int \cos(u)\,du = e^x + \frac{\sin(x^2)}{2} + C,$$

where we made the substitution $u = x^2$ (so $du = 2dx$).
Since $g(0) = 4$, we have $g(0) = e^0 + \frac{\sin(0)}{2} + C = 1 + C$, so we must take $C = 3$.

$$g(x) = e^x + \frac{\sin(x^2)}{2} + 3.$$