Math 125 Solutions to Midterm 2 (Inigo Montoya)

1. For each of the functions $f(x)$ given below, find $f'(x)$).

$\boxed{\text{4 points}}$ (a) $f(x) = \dfrac{1+2x^2}{1+x^5}$

Solution: This is a straightforward quotient rule problem:

$$f'(x) = \frac{(4x)(1+x^5) - (1+2x^2)(5x^4)}{(1+x^5)^2} = \frac{4x - 5x^4 - 6x^6}{(1+x^5)^2}$$

The simplification is not required.

$\boxed{\text{4 points}}$ (b) $f(x) = \sin(3x)\tan(x)$

Solution: Apply the product rule, with a chain rule for the $\sin(3x)$ term to get

$$f'(x) = 3\cos(3x)\tan(x) + \sin(3x)\sec^2(x).$$

$\boxed{\text{4 points}}$ (c) $f(x) = \arctan\left(\sqrt{1+4x}\right)$

Solution: Applying the chain rule, we get

$$\frac{1}{1+\left(\sqrt{1+4x}\right)^2} \cdot \frac{1}{2}(1+4x)^{-1/2} \cdot (4) = \frac{2}{(2+4x)\sqrt{1+4x}}$$

2. Compute each of the following derivatives as indicated:

$\boxed{\text{4 points}}$ (a) $\dfrac{d}{du}\left[\dfrac{u^3}{2} + \dfrac{2}{u^3}\right]$

Solution: Write this as $\frac{1}{2}u^3 + 2u^{-3}$ and apply the power rule to get

$$\frac{3}{2}u^2 - 6u^{-4}.$$

$\boxed{\text{4 points}}$ (b) $\dfrac{d}{dx}\left[\, e^x - \pi^4 \,\right]$

Solution: Remember that π^4 is a constant and so its derivative is zero. Thus, we have $\dfrac{d}{dx}\left[\, e^x - \pi^4 \,\right] = e^x$.

4 points (c) $\dfrac{d}{dw}\left[\sqrt{1+\sqrt{1+w}}\right]$

Solution: View this as $\frac{d}{dw}\left[\left(1+(1+w)^{1/2}\right)^{1/2}\right]$ and apply the chain rule:

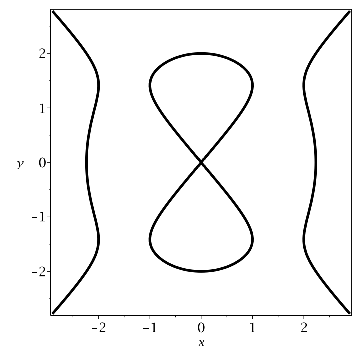
$$\frac{1}{2}\left(1+(1+w)^{1/2}\right)^{-\frac{1}{2}}\cdot\frac{1}{2}(1+w)^{-\frac{1}{2}}=\frac{1}{4\sqrt{1+w}\,\sqrt{1+\sqrt{1+w}}}$$

12 points 3. The set of points (x,y) which satisfy the relationship

$$y^2(y^2-4)=x^2(x^2-5)$$

lie on what is known as a "devil's curve", shown at right.

Write the equation of the line tangent to the given devil's curve at the point $(\sqrt{5},2)$.



Solution:

First, we use implicit differentiation to determine the slope of the tangent line. This will be slightly easier if we rewrite the equation as $y^4-4y^2=x^4-5x^2$ first. Differentiating with respect to x gives

$$4y^3y'-4\cdot 2y\cdot y'=4x^3-5\cdot 2x \quad \text{and so} \quad y'=\frac{x(2x^2-5)}{y(2y^2-4)}.$$

At our desired point, $x=\sqrt{5}$ and $y=2$, and so the slope is

$$y'=\frac{\sqrt{5}\cdot 5}{2\cdot 4}=\frac{5\sqrt{5}}{8}.$$

This means the desired line is

$$y-2=\frac{5\sqrt{5}}{8}(x-\sqrt{5}).$$

4. Let $f(x) = x\ln(3x)$

4 points (a) Calculate $f'(x)$

Solution: Applying the product rule (and the chain rule) gives

$$f'(x) = \ln(3x) + x\frac{1}{3x} \cdot 3 = \ln(3x) + 1.$$

4 points (b) Calculate $f''(x)$

Solution: Taking the derivative of the above, we get $f''(x) = \dfrac{1}{x}$.

3 points (c) For what values of x is $f(x)$ increasing?

Solution: As we all know, $f(x)$ is increasing when $f'(x) > 0$. Thus, using our answer from part (a) tells us that we need to know when

$$\ln(3x) + 1 > 0 \qquad \text{or, equivalently,} \qquad \ln(3x) > -1.$$

Exponentiating both sides gives $3x > e^{-1}$, so we know that

$$f(x) \text{ is increasing for } x > \frac{1}{3e}.$$

3 points (d) For what values of x is $f(x)$ concave down?

Solution: We need to determine when $f''(x) < 0$. From part (b), this means

$$\frac{1}{x} < 0 \qquad \text{that is,} \qquad x < 0.$$

However, remember that $\ln(4x)$ is only defined for $x > 0$. Thus $f(x)$ is concave up for all values of x in its domain. There are no values of x where $f(x)$ is concave down.

12 points 5. Give the x and y coordinates of the (absolute) maximum and minimum values of the function

$$y = x^4 - 8x^2 - 2 \qquad \text{where} \qquad -1 \le x \le 3.$$

Solution: First, we locate the critical points. Since the function is a polynomial, $f'(x)$ is defined everywhere, so we only need concern ourselves with the x for which $f'(x) = 0$.

Since $f'(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2)$, we have the critical points

$$x = 0 \qquad x = 2 \qquad x = -2$$

However, since we are concerned only with $-1 \leq x \leq 3$, we discard $x = -2$.

Now we evaluate f at each of the critical points, and the endpoints:

- $f(0) = -2$.
- $f(2) = 16 - 32 - 2 = -18$.
- $f(-1) = 1 - 8 - 2 = -9$.
- $f(3) = 81 - 72 - 2 = 7$.

The largest value of the above occurs at $x = 3$, $y = 7$. This is our absolute maximum.

The smallest occurs when $x = 2$ and $y = -18$, which is our absolute minimum.

$\boxed{\text{12 points}}$ 6. Calvin's family is visiting a winery in Cutchogue, and he wanders off into the fermenting room and dives into one of the large cylindrical† wine vats. The vat has a diameter of 6 feet and is 8 feet tall. The vinter hears the splash and quickly opens the taps to drain the vat, which drains at a rate of 5 cubic feet per minute. How quickly is the height of wine in the tank dropping when the wine is 4 feet deep?

†The volume of a cylinder of height h and radius r is $\pi r^2 h$ and its surface area (excluding top and bottom) is $2\pi rh$. The density of the wine is about .98 kg/L or 61 pounds per cubic foot. 5 cubic feet is about 38 gallons or 142 liters. The wine is a rather sweet Riesling, but is probably less sweet after Calvin has been in it.

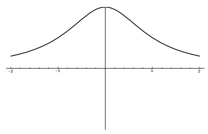
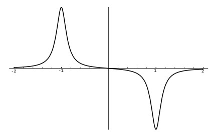
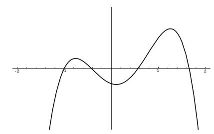
Solution: We have the formula for the volume of a cylinder $V = \pi r^2 h$. In our case, $r = 3$ since the diameter is 6, so we have $V = 9\pi h$ We want to know dh/dt.

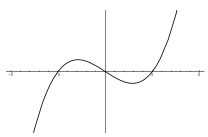
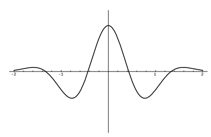
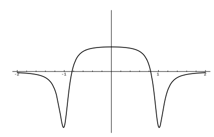
Since the vat is draining at a rate of 5 cubic feet per minute, we have $dV/dt = 5$.

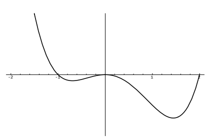
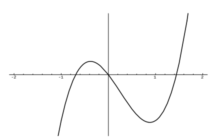
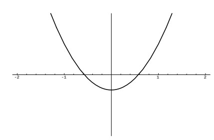
Using implicit differentiation, we get $\frac{dV}{dt} = 9\pi \frac{dh}{dt}$. So, we see that

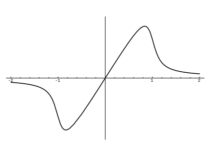
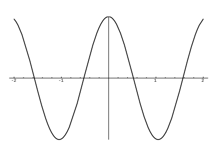
$$\frac{5}{9\pi} = \frac{dh}{dt}.$$

12 points 7. For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box. If the graph does not occur, use the letter **X**.

H  **A:**  **B:** 

F  **C:**  **D:** 

E  **E:**  **F:** 

D  **G:**  **H:**