MATH 125 Solutions to First Midterm

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

   (a) 3 points $\lim_{x\to 0} \frac{\sin x}{\tan x}$

   Solution:

$$\lim_{x\to 0} \frac{\sin x}{\tan x} = \lim_{x\to 0} \frac{\sin x}{\frac{\sin x}{\cos x}} = \lim_{x\to 0} \cos x = \cos(0) = 1$$

   (b) 3 points $\lim_{x\to +\infty} \frac{5x^2 - 4x - 1}{x^2 - 1}$

   Solution:

$$\lim_{x\to +\infty} \frac{5x^2 - 4x - 1}{x^2 - 1} = \lim_{x\to +\infty} \frac{5x^2}{x^2} = \lim_{x\to +\infty} 5 = 5.$$

   (c) 3 points $\lim_{x\to +\infty} \sqrt{4x^2 + x} - 2x$

   Solution:

$$\begin{aligned}
\lim_{x\to +\infty} \sqrt{4x^2 + x} - 2x &= \lim_{x\to +\infty} \left(\sqrt{4x^2 + x} - 2x\right) \frac{\sqrt{4x^2+x} + 2x}{\sqrt{4x^2+x} + 2x} \\
&= \lim_{x\to +\infty} \frac{4x^2 + x - 4x^2}{\sqrt{4x^2+x} + 2x} \\
&= \lim_{x\to +\infty} \frac{x}{\sqrt{4x^2+x} + 2x} \\
&= \lim_{x\to +\infty} \frac{\frac{1}{x}}{\frac{1}{x}} \cdot \frac{x}{\sqrt{4x^2+x} + 2x} \\
&= \lim_{x\to +\infty} \frac{1}{\sqrt{4 + \frac{1}{x}} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}
\end{aligned}$$

   (d) 3 points $\lim_{x\to 0^-} \frac{1}{x^5}$

   Solution: We are only considering $x < 0$, so $1/x^5$ is always negative. As x approaches 0 from the left, $1/x^5$ gets larger and larger in absolute value. Hence $\lim_{x\to 0^-} \frac{1}{x^5} = -\infty$

(e) [3 points] $\lim_{x\to 0} \frac{(2+x)^2-4}{x}$

Solution:

$$\lim_{x\to 0} \frac{(2+x)^2-4}{x} = \lim_{x\to 0} \frac{(4+4x+x^2)-4}{x} = \lim_{x\to 0} \frac{4x+x^2}{x} = \lim_{x\to 0} 4+x = 4$$

You could also do this by noticing that this is the definition of $f'(2)$ where $f(x) = x^2$ and use the power rule to see that $f'(x) = 2x$, so $f'(2) = 4$, but I doubt anyone did that.

2. [6 points] Let

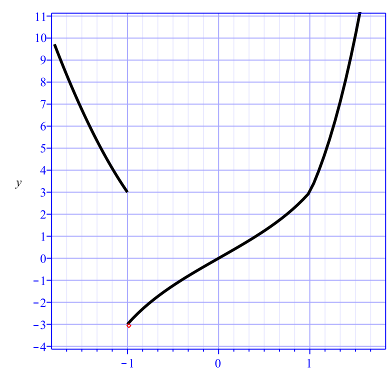
$$f(x) = \begin{cases} 3x^2 & \text{if } x < -1, \\ 3\tan(\frac{\pi}{4}x) & \text{if } -1 \le x \le 1, \\ 3x^3 & \text{if } x > 1. \end{cases}$$

For which values of x is $f(x)$ continuous? Justify your answer.

Solution: At right is the graph of $f(x)$.
Since $3x^2$, $3\tan(\frac{\pi}{4}x)$, and $3x^3$ are all continuous on their respective domains, we only need to check whether they match up at -1 and 1. That is, we need to see whether

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) \quad \text{or} \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x).$$

At $x = -1$, we see that $3(-1)^2 = 3$ and $3\tan(-\frac{\pi}{4}) = -3$, so f **is not** continuous at -1. But at $x = +1$, we have $3(1)^3 = 3$ and $3\tan(\frac{\pi}{4}) = 3$, and so f **is** continuous at 1.
Thus, f is continuous for all real numbers except $x = -1$.



3. Let $f(x) = 2x^3 - 4x + 4$.

(a) [5 points] Find $f'(1)$.

Solution: Using the power rule, $f'(x) = 6x^2 - 4$, so $f'(1) = 2$.

(b) [5 points] Write the equation of the line tangent to $f(x)$ at the point $P = (1,2)$.

Solution: We just neet the equation of the line of slope 2 passing through the point $(1,2)$. This is

$$y - 2 = 2(x-1) \qquad \text{or} \qquad y = 2x$$

4. [6 points] Write a limit that represents the slope of the graph

$$y = \begin{cases} 8 + x\ln|x| & x \neq 0 \\ 8 & x = 0 \end{cases}$$

at $x = 0$. You **do not need to evaluate the limit.**

Solution: To do this, we need to remember the definition of the derivative, which is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

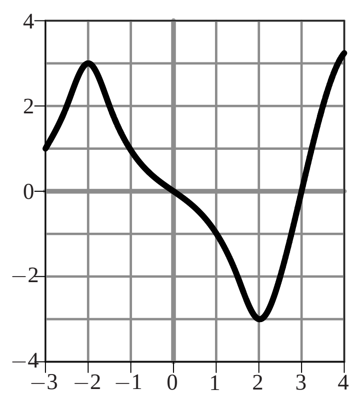
In the current case, $a = 0$, so $f(a+h) = f(h)$. Notice that $f(0) = 8$, so we have

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{(8 + h\ln|h|) - 8}{h}$$

This simplifies to

$$\lim_{h\to 0} \frac{h\ln|h|}{h} = \lim_{h\to 0} \ln|h| = -\infty,$$

although it wasn't required for you to do this.



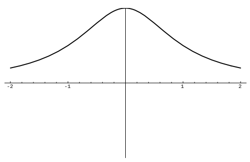
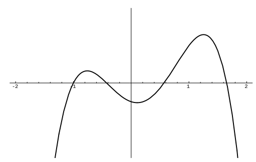
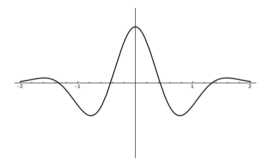
5. At right is the graph of **the derivative** f' of a function.
   (a) [4 points] List all values of x with $-3 \leq x \leq 4$ where $f(x)$ has a local maximum.

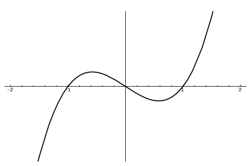
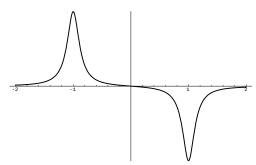
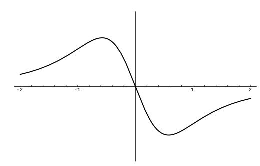
   Solution: A local maximum for $f(x)$ will occur where $f'(x)$ changes from positive to negative. This happens at $x = 0$.

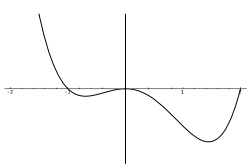
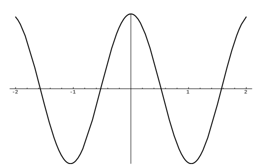
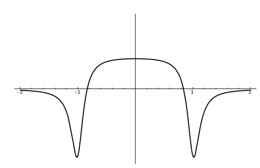
   (b) [4 points] At $x = -1$, is $f(x)$ concave up, concave down, or neither?

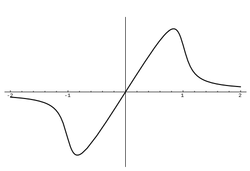
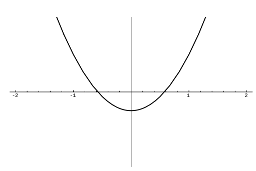
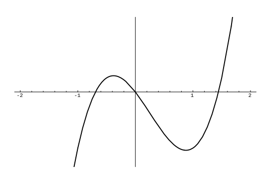
   Solution: We know that a function is concave up when its second derivative is positive, and concave down when f'' is negative. The graph shows $f'(x)$, which is decreasing near $x = -1$. That means the derivative of $f'(x)$ is negative near $x = -1$, so $f''(-1) < 0$. Hence $f(x)$ is concave down at $x = -1$.

6. 16 points For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box.

D  **A:**  **B:** 

G  **C:**  **D:** 

H  **E:**  **F:** 

F  **G:**  **H:** 

7. Let $f(x) = \dfrac{4-x^2}{(3+x)^2}$.

   (a) 4 points Identify the horizontal asymptotes of $f(x)$. If there are none, write "NONE".

   Solution: To find the horizontal asymptotes, we calculate the limit as $x \to \infty$. Thus,

$$\lim_{x\to\infty} \frac{4-x^2}{(3+x)^2} = \lim_{x\to\infty} \frac{4-x^2}{9+6x+x^2} = \lim_{x\to\infty} \frac{-x^2}{x^2} = -1$$

   So there is a horizontal asymptote at $y = -1$.

   (b) 4 points Identify the vertical asymptotes of $f(x)$. If there are none, write "NONE".

   Solution: There will be a vertical asymptote whenever there is a finite value $x = a$ such that $f(x) \to \pm\infty$ as $x \to a$ (this could be a one-sided limit). This happens when the denominator is zero but the numerator is non-zero.

   For this function, the denominator is zero when $x = -3$, and so we have a vertical asymptote at $x = -3$, that is,

$$\lim_{x\to -3} \frac{4-x^2}{(3+x)^2} = -\infty.$$

8. 8 points An exponential function of the form $y = Ca^x$ passes through the points $(1,6)$ and $(3,24)$. Find C and a.

   Solution: Since the function passes through $(1,6)$ and $(3,24)$, we know that

$$6 = Ca^1 \qquad \text{and} \qquad 24 = Ca^3$$

   From the first equation, we know that $C = 6/a$, and putting this into the second equation, we have $24 = (6/a)a^3$, or $4 = a^2$. Thus $a = 2$. (We must have $a > 0$, or a^x doesn't make sense.)

   Since $a = 2$, we have $C = 6/2 = 3$.

   Thus, the function is $y = 3 \cdot 2^x$