

1. For each of the functions $f(x)$ given below, find $f'(x)$.

(a) 3 points $f(x) = x^9 + 5x^4 + 2x^2 + \pi^2$

Solution: Don't forget that $\pi^2 \approx 9.87$, so its derivative is 0. Thus $f'(x) = 9x^8 + 20x^3 + 4x$.

(b) 3 points $f(x) = \cos(x) \sin(4x)$

Solution: This use the product rule and the chain rule:

$$f'(x) = -\sin(x) \sin(4x) + 4 \cos(x) \cos(4x)$$

(c) 3 points $f(x) = \frac{\sin(x)}{\cos(x)}$

Solution: After simplifying $\frac{\sin(x)}{\cos(x)} = \tan(x)$, we just remember that the derivative of $\tan(x)$ is $\sec^2(x)$.

Alternatively, if you prefer to use the quotient rule, you should get

$$\frac{\cos(x) \cos(x) + \sin(x) \sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

(d) 3 points $f(x) = \arctan(x^2)$

Solution: This is a straight chain-rule problem.

$$f'(x) = \frac{1}{1 + (x^2)^2} \cdot (2x) = \frac{2x}{1 + x^4}$$

Several people were confused and thought that $\arctan(x) = \frac{1}{\tan(x)} = \cot(x)$; this is nonsense. You should know that $\arctan(x) = y$ means that $\tan(y) = x$.

2. Compute each of the following derivatives as indicated:

(a) 3 points $\frac{d}{dt} \left[\frac{e^t - e^{-t}}{e^t + e^{-t}} \right]$

Solution: Applying the quotient rule gives us

$$\begin{aligned} \frac{(e^t + e^{-t})(e^t + e^{-t}) - (e^t - e^{-t})(e^t - e^{-t})}{(e^t + e^{-t})^2} &= \frac{(e^{2t} + 2 + e^{-2t}) - (e^{2t} - 2 + e^{-2t})}{(e^t + e^{-t})^2} \\ &= \frac{4}{(e^t + e^{-t})^2} \end{aligned}$$

(b) 3 points $\frac{d}{du} [u \ln(u)]$

Solution: Using the product rule, we have $1 \cdot \ln(u) + u \cdot \frac{1}{u} = \ln(u) + 1$

(c) 3 points $\frac{d}{dz} [\ln(\sec(3z))]$

Solution: From the chain rule, we have

$$\frac{1}{\sec(3z)} \cdot \sec(3z) \tan(3z) \cdot 3 = 3 \tan(3z)$$

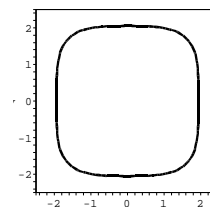
(d) 3 points $\frac{d}{dx} [e^x - x^e]$

Solution: Remembering that e is a constant, we have $e^x - e x^{e-1}$.

3. 10 points Let \mathcal{C} be the curve which consists of the set of points for which

$$x^4 + x^2 + y^4 = 18$$

(see the graph at right).



Write the equation of the line tangent to \mathcal{C} which passes through the point $(1, -2)$.

Solution: In order to write the equation of a line, we need a point on the line (which we have: $(1, -2)$) and the slope of the line. For the slope, we need dy/dx at the given point.

We *could* solve for y , getting $y = \pm \sqrt[4]{18 - x^4 - x^2}$, and take the derivative of the resulting function to get $y' = \pm (18 - x^4 - x^2)^{-3/4} (-4x^3 - x)$.

Instead, let's use implicit differentiation:

$$4x^3 + 2x + 4y^3 \frac{dy}{dx} = 0$$

Since we want the slope when $x = 1$ and $y = -2$, we plug in and solve for dy/dx .

$$4 + 2 - 32 \frac{dy}{dx} = 0, \quad \text{so} \quad \frac{dy}{dx} = \frac{-6}{-32} = \frac{3}{16}$$

Thus, the desired line is

$$y+2 = \frac{3}{16}(x-1) \quad \text{or} \quad y = \frac{3}{16}x - \frac{35}{16}$$

This next problem uses material we didn't cover yet. Such things won't be on our midterm.

4. 10 points Give the x and y coordinates of the (absolute) maximum and minimum values of the function

$$y = x^4 - 8x^2 + 1 \quad \text{where} \quad -1 \leq x \leq 3.$$

Solution: First, we locate the critical points. Since the function is a polynomial, $f'(x)$ is defined everywhere, so we only need concern ourselves with the x for which $f'(x) = 0$.

Since $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2)$, we have the critical points

$$x = 0 \quad x = 2 \quad x = -2$$

However, since we are concerned only with $-1 \leq x \leq 3$, we discard $x = -2$.

Now we evaluate f at each of the critical points, and the endpoints:

- $f(0) = 1$.
- $f(2) = 16 - 32 + 1 = -15$.
- $f(-1) = 1 - 8 + 1 = -6$.
- $f(3) = 81 - 72 + 1 = 10$.

The largest value of the above occurs at $x = 3$, $y = 10$. This is our absolute maximum.

The smallest occurs when $x = 2$ and $y = -15$, which is our absolute minimum.

5. Let $f(x) = x e^{-4x}$.

- (a) 3 points Calculate $f'(x)$

Solution: We use the product rule and the chain rule:

$$f'(x) = e^{-4x} - 4x e^{-4x}$$

- (b) 3 points Calculate $f''(x)$

Solution: Taking the derivative of the above gives

$$f''(x) = -4e^{-4x} - 4e^{-4x} + 16x e^{-4x}$$

which simplifies to

$$16x e^{-4x} - 8e^{-4x}$$

- (c) 3 points For what values of x is $f(x)$ increasing?

Solution: To answer this, we need to know when $f'(x) > 0$, that is, where

$$e^{-4x} - 4x e^{-4x} > 0$$

Factoring out the exponential term gives $e^{-4x}(1 - 4x) > 0$, and since e^{-4x} is always positive, we only need ask where $1 - 4x > 0$. This happens for

$$x < \frac{1}{4}.$$

- (d) 3 points For what values of x is $f(x)$ concave down?

Solution: We need to know when $f''(x) < 0$, so factor $f''(x)$ as

$$8e^{-4x}(2x - 1).$$

As before, we can ignore the exponential term, since it is always positive, and we see that $f''(x) < 0$ when $x < 1/2$.

This next problem uses material we didn't cover yet. Such things won't be on our midterm.

6. 10 points A leaky oil tanker is anchored offshore. Because the water is very calm, the oil slick always stays circular as it expands, with a uniform depth of 1 meter. If the oil is leaking from the tanker at a rate of $100 \frac{m^3}{hr}$, how fast is the radius of the slick expanding (in $\frac{m}{hr}$) when the diameter is 20 meters?

Solution: First, notice that since we are given the rate of oil leaking out from the tanker, this is the rate of change of volume of oil ($dV/dt = 100 \frac{m^3}{hr}$), and we want to know the rate of change of the radius (dr/dt). This means we need to write a formula for the volume of the oil as a function of the radius.

Since we are told that the oil slick is circular and has a constant depth of 1 meter, it is a cylinder of height 1 and radius r . That is, $V = \pi r^2$

Taking the derivative with respect to time gives $\frac{dV}{dt} = 2\pi r \frac{dr}{dt}$ and plugging in the given values yields

$$100 = 2\pi \cdot 10 \frac{dr}{dt},$$

so we have

$$\frac{dr}{dt} = \frac{100}{20\pi} = \frac{5}{\pi}.$$