Solutions to review problems MAT 125, Fall 2004

1. For each of the following functions, find the absolute maximum and minimum values for $f(x)$ in the given intervals. Also state the x value where they occur.

a. $\dfrac{\ln x}{x}$ for $1/3 \le x \le 3$.

Solution: First, we look for critical points.

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2}$$

This means there will be a critical point when the numerator is zero (that is, when $\ln x = 1$, or $x = e$) or when the denominator is zero (at $x = 0$, which is not in the domain).

Now we check which is the biggest:

$$f\left(\frac{1}{3}\right) = \frac{\ln 1/3}{1/3} = \frac{-\ln 3}{1/3} = -3\ln 3$$

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

$$f(3) = \frac{\ln 1/3}{1/3} = \frac{\ln 3}{3}$$

Of these choices, we see that the smallest value is the negative one, so the absolute minimum occurs at $(1/3, -3\ln 3)$, and the largest is at the critical point, so the absolute maximum is at $(e, 1/e)$.

b. $e^x - x$ for $-1 \le x \le 1$.

Solution: The critical points occur when the derivative is zero, that is, when $e^x - 1 = 0$. That means the only critical point is $x = 0$.

Checking the endpoints and the critical points, we see

$$f(-1) = e^{-1} - 1$$

$$f(0) = e^0 - 0 = 1$$

$$f(1) = e - 1$$

Since $f(-1) < f(1) < f(0)$, we have the absolute minimum at $(-1, 1/e - 1)$, and the absolute maximum at $(0, 1)$.

c. $x - \ln x$ for $1/2 \leq x \leq 2$.

Solution: $f'(x) = 1 - 1/x$, so there is a critical point only when $x = 1$. Checking the choices, we have

$$f\left(\frac{1}{2}\right) = \frac{1}{2} - \ln\frac{1}{2} = \frac{1}{2} + \ln 2$$
$$f(1) = 1 - \ln 1 = 1$$
$$f(2) = 2 - \ln 2$$

Since $\ln 2 \approx 0.7$, the absolute maximum is at $(1/2, 1/2 + \ln 2)$, and the absolute minimum occurs at $(1, 1)$.

d. $\sin(\cos(x))$ for $0 \leq x \leq 2\pi$.

Solution: $f'(x) = \cos(\cos(x)) \cdot (-\sin(x))$, so there will be a critical point when either $\sin(x) = 0$ or $\cos(\cos(x)) = 0$. The first is easy, giving us $x = 0$ or $x = \pi$. For the second choice, we have to work a little more:

If $\cos(\cos(x)) = 0$, then it must be that $\cos(x)$ is either $\pi/2$ or $-\pi/2$, since these are the only two angles that have a cosine of 0. But $\pi/2 > 1$, so it is not possible to have x so that $\cos(x) = \pi/2$.

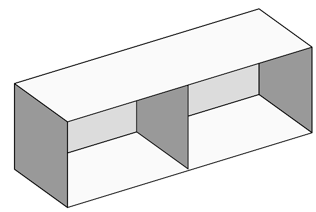
Since we have our critical points in hand, we can now finish the problem by checking out our choices:

$$f(0) = \sin(\cos(0)) = \sin(1)$$
$$f(\pi) = \sin(\cos(\pi)) = \sin(-1) = -\sin(1)$$
$$f(2\pi) = \sin(\cos(2\pi)) = \sin(1)$$

If we had a calculator, we could check what $\sin(1)$ is approximately equal to. But this isn't necessary, because we really only need to know if it is positive or negative. But we know that $\sin(x) > 0$ for $0 < x < \pi$, and $0 < 1 < \pi$, so $\sin(1) > 0$. This means our absolute maximum occurs at both endpoints, namely $(0, \sin(1))$ and $(2\pi, \sin(1))$. The absolute minimum occurs right in the middle, at $(\pi, -\sin(1))$.

By the way, $\sin(1) \approx 0.84147$.

2. An open, divided box is to be constructed from three square pieces of wood and three rectangular ones. The rectangular pieces will be used for the top, bottom and back, while the squares will form the ends and the divider. If the total area of the wood to be used is 9 sq. ft., what are the dimensions which will maximize the volume of the box?



Solution: We want to maximize the volume of the box, which is given by $V = l \cdot w \cdot h$, where the l, w, and h are the length, width, and height of the box, respectively.

First, since the ends are square, this means $w = h$, so $V = lh^2$.

Now, we only have 9 sq. ft. of wood to work with, so this will give us some relationship between l and h. We have the box made from three square pieces that are $h \times h$, and three rectangular pieces that are $l \times h$. So,

$$9 = 3h^2 + 3lh$$

Solving for l gives

$$\frac{3 - h^2}{h} = l$$

.

This means that we can write the volume as a function of h alone:

$$V(h) = \frac{3 - h^2}{h} h^2 = 3h - h^3$$

We want to look for the critical points of V.

$V'(h) = 3 - 3h^2$, so we have a critical point when $3(1 - h)(1 + h) = 0$, that is, when $h = 1$ or $h = -1$. It is hard to have a box with a negative height, so the only relevant critical point is $h = 1$.

We need to check if this is a minimum or a maximum. Naturally, we suspect it is a maximum, but let's be sure. We can use the second derivative test:

$$V''(h) = -6h$$

and so

$$V''(1) = -6$$

Since $V''(1) < 0$, the function is concave down there, so $h = 1$ is a local maximum.

If $h = 1$, then $l = 2$ and $w = 1$, so the box should have dimensions $1 \times 1 \times 2$ to have the maximal volume, which is 2 cubic feet.

3. For the functions $f(x) = xe^{-\frac{x^2}{2}}$ and $g(x) = e^{\frac{6x-9x^2}{4}}$ do the following:

a. Compute the first and second derivatives.

Solution:

$$\begin{aligned} f'(x) &= e^{-\frac{x^2}{2}} - x^2 e^{-\frac{x^2}{2}} && = e^{-\frac{x^2}{2}}(1-x^2) \\ f''(x) &= -xe^{-\frac{x^2}{2}}(1-x^2) + e^{-\frac{x^2}{2}}(-2x^2) && = e^{-\frac{x^2}{2}}(-3x+x^3) \end{aligned}$$

$$\begin{aligned} g'(x) &= e^{\frac{6x-9x^2}{4}}\left(\frac{6-18x}{4}\right) && = e^{\frac{6x-9x^2}{4}}\left(\frac{3-9x}{2}\right) \\ g''(x) &= e^{\frac{6x-9x^2}{4}}\left(\frac{3-9x}{2}\right)^2 - \frac{9}{2}e^{\frac{6x-9x^2}{4}} && = \left(\frac{-9-54x+81x^2}{4}\right)e^{\frac{6x-9x^2}{4}} \end{aligned}$$

b. Find the intervals on which the function is increasing and decreasing.

Solution: Remember that a function increases when the derivative is positive, and decreases when it is negative. So, $f(x)$ will be increasing when $e^{-\frac{x^2}{2}}(1-x^2) > 0$. Since $e^y > 0$ for all y, this means we must have $1 - x^2 > 0$. That, in turn, tells us that $f(x)$ is increasing when $-1 < x < 1$. Similarly, f decreases when $x < -1$ or when $x > 1$.

For g, we need $g'(x) > 0$, and so we have $(3-9x)/2 > 0$, that is $x < 1/3$. Hence g is increasing when $x < 1/3$, and decreasing when $x > 1/3$.

c. Locate all local maximima and minima.

Solution: Both $f(x)$ and $g(x)$ are continuous for all x, so we only have to find where $f'(x) = 0$ and $g'(x) = 0$.

For f, if $f'(x) = 0$, then from the previous part we know $x = -1$ and $x = 1$ are the critical points. From the previous part, since f changes from decreasing to increasing at $x = -1$, we know that $(-1, -1/\sqrt{e})$ is a relative minimum. Similarly, since f changes from increasing to decreasing at $x = 1$, we know that $(1, 1/\sqrt{e})$ is a relative maximum. If you prefer, you make the same conclusion using the second derivative test, since $f''(-1) > 0$ and $f''(1) < 0$.

For g, again using the previous part we see that $x = 1/3$ is the only critical point. So $g(x)$ achieves its maximum at $(1/3, e^{1/4})$, since it goes from increasing to decreasing there.

d. Locate all inflection points.

Solution: We have to look for the places where $f(x)$ and $g(x)$ change concavity, so we look where the second derivative is zero.

$$f''(x) = e^{-\frac{x^2}{2}}(-3x + x^3) = e^{-\frac{x^2}{2}}(x)(x + \sqrt{3})(x - \sqrt{3}),$$

so $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$. We see that all three are inflection points, since $f''(x) < 0$ for $x < -\sqrt{3}$, $f''(x) > 0$ for $-\sqrt{3} < x < 0$, $f''(x) < 0$ for $0 < x < \sqrt{3}$, and $f''(x) > 0$ for $x > \sqrt{3}$.

Similarly,

$$g''(x) = \left(\frac{-9 - 54x + 81x^2}{4}\right) e^{\frac{6x-9x^2}{4}} = \frac{9}{4}\left(9x^2 - 6x - 1\right) e^{\frac{6x-9x^2}{4}}$$

which is zero exactly when $9x^2 - 6x - 1 = 0$. Using the quadratic formula, we get

$$x = \frac{1 \pm \sqrt{2}}{3}$$

as the inflection points of g.

e. For which x is the function concave up?

Solution: Since a function is concave up exactly when its second derivative is positive, we know from the previous part that f is concave up when $-\sqrt{3} < x < 0$ or when $x > \sqrt{3}$.

Similarly, we know g is concave up for $x < (1 - \sqrt{2})/3$ and for $x > (1 + \sqrt{2})/3$.

f. Find all horizontal and vertical asymptotes, if any exist.

Solution: Since f and g are both continuous, neither has any vertical asymptotes.

For horizontal asymptotes, we take limits as $x \to \pm\infty$:

$$\lim_{x\to\pm\infty} xe^{-x^2/2} = \lim_{x\to\pm\infty} \frac{x}{e^{x^2/2}} = \lim_{x\to\pm\infty} \frac{1}{xe^{x^2/2}} = \frac{1}{\text{huge}} = 0$$

where we used L'Hôpital's rule above since the limit was of the form $0 \cdot \infty$. So $f(x)$ has a horizontal asymptote of 0 in both the positive and negative directions.

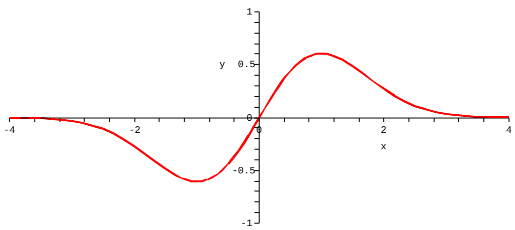
For $g(x)$, we don't need L'Hôpital's rule:

$$\lim_{x\to\pm\infty} e^{\frac{6x-9x^2}{4}} = e^{\lim_{x\to\pm\infty}\frac{6x-9x^2}{4}} = e^{-\infty} = 0.$$

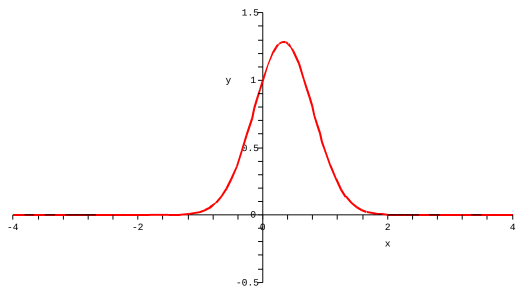
So, just like f, g also has 0 as a horizontal asymptote in both directions.

g. Use the above information to sketch the graph of the function.

Solution:



f(x)

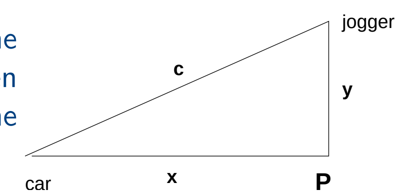


g(x)

4. A man starts running north at 10 km/h from a point P. At the same time, a car is 50 km. west of P and travels on a straight road directly east (towards P) at 60 km/h. How fast is the distance between the jogger and the driver changing 30 minutes later? Is the distance between them increasing or decreasing?

Solution:

As in the figure at right, let x represent the distance between the car and P after t hours, and let y represent the distance between the jogger and P after t hours. Let c be the distance between the jogger and the car. What we want is $\frac{dc}{dt}$ when $t = 1/2$.



We also know that $\frac{dy}{dt} = +10$ and $\frac{dx}{dt} = -60$ (this is negative because the car is moving towards P, so the distance is decreasing.

By the Pythagorean theorem, we have that

$$c^2 = x^2 + y^2$$

Differentiating gives us

$$2c\frac{dc}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Since the car started 50 km away from P and traveled for half an hour at 60 km/hr, at the time of interest, $x = 50 - 30 = 20$ km, and since the jogger ran for half an hour, we have $y = 5$. This means $c = \sqrt{20^2 + 5^2} = \sqrt{425} = 5\sqrt{17}$. Plugging all this into the relation above gives us

$$2 \cdot 5\sqrt{17}\,\frac{dc}{dt} = 2 \cdot 20 \cdot (-60) + 2 \cdot 5 \cdot 10$$

so

$$\frac{dc}{dt} = -\frac{14}{\sqrt{17}} \approx -3.39\,\text{km/hr}$$

Since the rate of change is negative, the distance between them is decreasing.

5. Find the following limits, if they exist. If the limit fails to exist, distinguish between $+\infty$, $-\infty$, and no limiting behavior (DNE).

- $\displaystyle\lim_{x\to 0} \frac{2\sin(5+x) - 2\sin(5)}{x}$

  Solution: This is just $f'(5)$ where $f(x) = 2\sin(x)$. Since $f'(x) = 2\cos(x)$, the value of the limit is $2\cos(5)$.

- $\displaystyle\lim_{x\to 0} \ln[(e^x)^2] - \cos x$

  Solution: First, simplify $\ln[(e^x)^2] = 2\ln e^x = 2x$. Then the limit is easy, since everything is obviously continuous. Plugging in gives us $\displaystyle\lim_{x\to 0} 2x - \cos x = -\cos(0) = -1$.

- $\displaystyle\lim_{x\to+\infty} \frac{1}{2+\cos x}$

  Solution: As x gets large, $\cos x$ oscillates between 1 and -1, so the function varies between $\frac{1}{3}$ and 1. There is no limiting behavior.

- $\displaystyle\lim_{x\to+\infty} \ln(2x^2+1) - \ln(x^2-3)$

  Solution: As x gets large, we have the difference of two large numbers. So, we need to do some algebra to combine them. Using properties of the log, we rewrite this as

$$\lim_{x\to+\infty} \ln\left(\frac{2x^2+1}{x^2-3}\right) = \ln\left(\lim_{x\to+\infty} \frac{2x^2+1}{x^2-3}\right) = \ln\left(\lim_{x\to+\infty} \frac{2x^2}{x^2}\right) = \ln(2)$$

- $\lim_{x\to 3+} x^2 \ln(x-3)$

  Solution:

$$\lim_{x\to 3+} x^2 \ln(x-3) = 9 \lim_{x\to 3+} \ln(x-3) = -\infty$$

- $\lim_{x\to 0} x \sin(e^{1/x})$

  Solution: Since $-1 < \sin y < 1$ no matter what y is, we use the squeeze theorem to see

$$0 = \lim_{x\to 0} -x \le \lim_{x\to 0} x \sin(e^{1/x}) \le \lim_{x\to 0} x = 0$$

  so the desired limit is 0.

6. A manufacturer wants to make a cylindrical can that has a volume of 400 cm^3. What should the radius of the base and the height be in order to minimize the amount of material needed (that is, in order to minimize the surface area)?

What if the top and bottom of the can are cut from rectangular pieces, and you cannot use the remaining corners for anything else?

Solution: First, we do the problem assuming nothing will be wasted.

The volume of a cylinder is $V = \pi r^2 h$, where r is the radius of the cylinder, and h is the height. We are given that $V = 400$, so this means

$$400 = \pi r^2 h \quad \text{so we have} \quad h = \frac{400}{\pi r^2}.$$

Now, we want to mimize the surface area. Let S be the surface area, and so we have

$$S = \text{Area of top} + \text{Area of bottom} + \text{Area of cylindrical part}$$

The area of the top and bottom is easy: each is just a circle of radius r, so that is πr^2. To see what the area of the cylindrical part is, imagine cutting it down the side and opening it up. This will be a rectangle which is h tall and $2\pi r$ wide (we get $2\pi r$ because it wraps all the way around the circle which forms the top). The area of this is $2\pi r h$. So, we have

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{400}{\pi r^2} = 2\pi r^2 + \frac{800}{r}$$

To find the minimum, we look for a critical point.

$$S'(r) = 4\pi r - \frac{800}{r^2} = \frac{4\pi r^3 - 800}{r^2}$$

We see that we will get a critical point when $r = 0$ and when $4\pi r^3 = 800$. The latter happens when $r^3 = 200/\pi$, that is,

$$r = \sqrt[3]{\frac{200}{\pi}} \approx 3.993$$

We need to confirm that this is a minimum. One easy way is the second derivative test:

Since $S''(r) = 4\pi + \frac{1600}{r^3}$, we see that $S(r)$ is concave up for positive r, so the critical point we found must be a minimum.

With $r = \sqrt[3]{\frac{200}{\pi}}$, the height of the can should be

$$h = \frac{400}{\pi r^2} = \frac{400}{\pi} \frac{\pi^{\frac{2}{3}}}{200^{\frac{2}{3}}} = 2\sqrt[3]{\frac{200}{\pi}} \approx 7.989$$

Note that this makes a can with the same diameter as the height. If you want to play with this yourself (no calculus, just cylinders), see the applet at
http://www.math.tamu.edu/AppliedCalc/Classes/Optimization/cylinder.html

If instead we can't reuse the cut-off pieces of the tops, the surface area to minimize becomes

$$A(r) = 8r^2 + \frac{800}{r}$$

since the amount of metal needed to make the top and bottom is a pair of $2r \times 2r$ squares.

To find this critical point, we have

$$A'(r) = 16r - \frac{800}{r^2} = \frac{16r^3 - 800}{r^2}$$

This has critical points when $r = 0$ or $r = \sqrt[3]{50} \approx 3.684$. As before, the second derivative test shows the positive solution to be a minimum. In this case, $h = \frac{400}{\pi \sqrt[3]{50^2}} \approx 9.381$. The resulting can is taller and skinnier than in the first case. (If you think about it, this makes sense, since due to the wasted material, the tops are more expensive to make than in the first case. To minimize the cost, you use less material for the tops and bottoms.)

7. Find antiderivatives for each of the following:

- $2x^7$

  Solution: If $f'(x) = 2x^7$, then $f(x) = x^8/4 + C$.

- $\dfrac{x^2+1}{x}$

  Solution: If $f'(x) = \dfrac{x^2+1}{x} = x + 1/x$, so $f(x) = x^2/2 + \ln|x| + C$.

- e^{2x}

  Solution: If $f'(x) = e^{2x}$, then $f(x) = e^{2x}/2 + C$.

- $\sin(3x) + 3\cos(x)$

  Solution: If $f'(x) = \sin(3x) + 3\cos(x)$, then $f(x) = -\cos(3x)/3 + 3\sin(x) + C$.

- $\sqrt{2x^3}$

  Solution: If $f'(x) = \sqrt{2x^3} = \sqrt{2}\,x^{3/2}$, then

$$f(x) = \frac{2\sqrt{2}}{5}x^{5/2} + C$$

- $\dfrac{5}{x^2+1}$

  Solution: If $f'(x) = \dfrac{5}{x^2+1}$, then $f(x) = 5\arctan(x) + C$.

8. A leaky oil tanker is anchored offshore. Because the water is very calm, the oil slick always stays circular as it expands, with a uniform depth of 1 meter. How rapidly is oil leaking from the tanker (in $\frac{m^3}{hr}$) if the radius of the slick is expanding at a rate of $8\frac{m}{hr}$ when the diameter is 20 meters?

Solution: Since the oil slick is circular, its area is given by $A = \pi r^2$, and since it has a uniform depth of 1 meter, the volume of oil is also $V = \pi r^2$. We want to know how rapidly the oil is leaking from the tanker, that is, how fast is the volume of oil changing. Hence, we want to know $\frac{dV}{dt}$ when $r = 10$ (since the radius is half the diameter).

So, we have

$$V = \pi r^2 \qquad \text{and so} \qquad \frac{dV}{dt} = 2\pi r \frac{dr}{dt}$$

Since the radius is expanding at 8 meters per hour when $r = 10$, we have

$$\frac{dV}{dt} = 2\pi(10)(8) = 160\pi,$$

so the tanker is losing about 500 cubic meters of oil an hour.

9. Use Newton's method on the function $g(x) = x^4 + 64x - 48$, starting with the initial approximation $x_1 = 2$, to find the third approximation x_3 to the solution of the equation $g(x) = 0$. Yes, I know $g(2)$ is a stupid initial guess (since $g(2) = 96$, not real close to 0). Give your answer to nine hundred billion decimal places. (Hint: if you need a calculator, you're probably doing it wrong, or you have trouble multiplying by 2.)

Solution: For Newton's method, we need to use

$$x - \frac{g(x)}{g'(x)} = x - \frac{x^4 + 64x - 48}{4x^3 + 64}$$

So, as asked, we start with $x_1 = 2$, and get

$$x_2 = 2 - \frac{2^4 - 64 \cdot 2 - 48}{4 \cdot 2^3 + 64} = 2 - \frac{16 + 128 - 48}{32 + 64} = 2 - 1 = 1$$

Now we use x_2 to get x_3:

$$x_3 = 1 - \frac{1 + 64 - 48}{4 + 64} = 1 - \frac{17}{68} = \frac{3}{4}$$

10. Joe Spacesuit is on a spacewalk floating freely outside the space station. He is at rest relative to the space station, and then turns on his jets, which give him an acceleration of 10 ft/sec^2. He accelerates at this constant rate for 2 seconds, and then shuts the jets off. If he started out 200 feet from the space station, how long does it take him to get there?

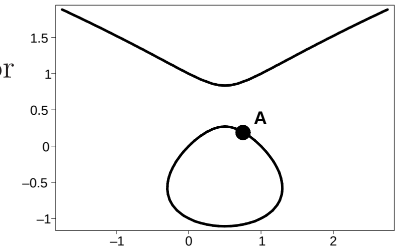
Solution: Joe's acceleration is given by $a(t) = 10$ for $0 \leq t \leq 2$, and $a(t) = 0$ for $t > 2$. His velocity is the antiderivative of $a(t)$, that is,

$$v(t) = 10t \quad \text{for } 0 \leq t \leq 2, \qquad v(t) = v(2) \quad \text{for } t > 2$$

We know $v(0) = 0$ since he was at rest relative to the station when he turned on the jets, and so we see that $v(2) = 20$.

Finally, his position is given by the antiderivative of the velocity: $s(t) = 5t^2$ for $t \leq 2$, and $s(t) = 20t + s(2)$ for $t > 2$.

We want to know when $s(t) = 200$. First, notice that $s(2) = 20$, so he travels the first 20 feet in 2 seconds. For the remaining 180 feet, he goes at a constant rate of 20 feet per second, so it takes him another 9 seconds. This means it takes $9 + 2 = 11$ seconds to cover the full 200 feet.



11. Consider the curve C which consists of the set of points for which

$$x^2 - x = y^3 - y$$

(see the graph at right).

a. Write the equation of the line tangent to C at the point $(1, 0)$.

Solution: We use implicit differentiation to get $2x - 1 = 3y^2y' - y'$. Plugging in $x = 1$ and $y = 0$ gives $2 - 1 = 0 - y'$, so

$$y' = -1 \qquad \text{at } (1, 0).$$

This means the tangent line is

$$y = -x + 1$$

b. Use your answer to part **a** to estimate the y-coordinate of the point with x-coordinate 3/4 marked A in the figure. Plug your estimate into the equation for C to determine how good it is.

Solution: When $x = 3/4$, we have $y = -3/4 + 1$ on the tangent line, so the point $(3/4, 1/4)$ should be close to the curve C.

If we plug in $x = \frac{3}{4}$ and $y = \frac{1}{4}$ to the original equation, we get $\dfrac{9}{16} - \dfrac{3}{4} \approx \dfrac{1}{64} - \dfrac{1}{4}$. If we write this all with a denominator of 64, we get

$$\frac{-12}{64} \approx \frac{-15}{64}$$

so we have an error of about $3/64$ or 0.047.