1. For each of the functions $f(x)$ given below, find $f'(x)$.

(a) **3 points** $f(x) = x^9 + 5x^4 + 4x^2 + \pi^2$

**Solution:** Don’t forget that $\pi^2 \approx 9.87$, so its derivative is 0.

$$f'(x) = 9x^8 + 20x^3 + 8x$$

(b) **3 points** $f(x) = \cos(x) \sin(3x)$

**Solution:** This requires the product rule, and the chain rule.

$$f'(x) = -\sin(x) \sin(3x) + 3 \cos(x) \cos(3x)$$

(c) **3 points** $f(x) = \frac{\sin(x)}{\cos(x)}$

**Solution:** After simplifying $\frac{\sin(x)}{\cos(x)} = \tan(x)$, we just remember that the derivative of $\tan(x)$ is $\sec^2(x)$.

Alternatively, if you prefer to use the quotient rule, you should get

$$\frac{\cos(x) \cos(x) + \sin(x) \sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

(d) **3 points** $f(x) = \arctan(x^3)$

**Solution:** This is a straight chain-rule problem.

$$f'(x) = \frac{1}{1 + (x^3)^2} \cdot (3x^2) = \frac{3x^2}{1 + x^6}$$

Several people were confused and thought that $\arctan(x) = \frac{1}{\tan(x)} = \cot(x)$; this is nonsense. You should know that $\arctan(x) = y$ means that $\tan(y) = x$.

2. Compute each of the following derivatives as indicated:

(a) **3 points** $\frac{d}{dt} \left[ \frac{e^t - e^{-t}}{e^t + e^{-t}} \right]$
Solution: Applying the quotient rule gives us
\[
\frac{(e^t + e^{-t})(e^t + e^{-t}) - (e^t - e^{-t})(e^t - e^{-t})}{(e^t + e^{-t})^2} = \frac{(e^{2t} + 2 + e^{-2t}) - (e^{2t} - 2 + e^{-2t})}{(e^t + e^{-t})^2} = \frac{4}{(e^t + e^{-t})^2}
\]

(b) 3 points  \( \frac{d}{du} [u \ln(u)] \)

Solution: Using the product rule, we have
\[
1 \cdot \ln(u) + u \cdot \frac{1}{u} = \ln(u) + 1
\]

(c) 3 points  \( \frac{d}{dz} [\ln(\sec(5z))] \)

Solution: From the chain rule, we have
\[
\frac{1}{\sec(5z)} \cdot \sec(5z) \tan(5z) \cdot 5 = 5 \tan(5z)
\]

(d) 3 points  \( \frac{d}{dx} [e^x - xe^x] \)

Solution: Remembering that \( e \) is a constant, we have \( e^x - e \cdot xe^{-1} \).

3. 10 points  Let \( C \) be the curve which consists of the set of points for which
\[
x^4 + x^2 + y^4 = 18
\]
(see the graph at right).
Write the equation of the line tangent to \( C \) which passes through the point \((-1, -2)\).

Solution: In order to write the equation of a line, we need a point on the line (which we have: \((-1, -2)\)) and the slope of the line. For the slope, we need \( dy/dx \) at the given point.
We could solve for \( y \), getting \( y = \pm \sqrt{18 - x^4 - x^2} \), and take the derivative of the resulting function to get \( y' = \pm (18 - x^4 - x^2)^{3/4}(-4x^3) \).
Instead, let’s use implicit differentiation:

\[ 4x^3 + 2x + 4y^3 \frac{dy}{dx} = 0 \]

Since we want the slope when \( x = -1 \) and \( y = -2 \), we plug in and solve for \( \frac{dy}{dx} \).

\[-4 - 2 - 32 \frac{dy}{dx} = 0, \quad \text{so} \quad \frac{dy}{dx} = \frac{6}{-32} = -\frac{3}{16} \]

Thus, the desired line is

\[ y + 2 = -\frac{3}{16}(x + 1) \quad \text{or} \quad y = -\frac{3}{16}x - \frac{35}{16} \]

4. **10 points** Give the \( x \) and \( y \) coordinates of the (absolute) maximum and minimum values of the function

\[ y = x^4 - 8x^2 - 2 \quad \text{where} \quad -1 \leq x \leq 3. \]

**Solution:** First, we locate the critical points. Since the function is a polynomial, \( f'(x) \) is defined everywhere, so we only need concern ourselves with the \( x \) for which \( f'(x) = 0 \).

Since \( f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2) \), we have the critical points

\[ x = 0 \quad x = 2 \quad x = -2 \]

However, since we are concerned only with \(-1 \leq x \leq 3\), we discard \( x = -2 \).

Now we evaluate \( f \) at each of the critical points, and the endpoints:

- \( f(0) = -2 \).
- \( f(2) = 16 - 32 - 2 = -18 \).
- \( f(-1) = 1 - 8 - 2 = -9 \).
- \( f(3) = 81 - 72 - 2 = 7 \).

The largest value of the above occurs at \( x = 3, y = 7 \). This is our absolute maximum. The smallest occurs when \( x = 2 \) and \( y = -18 \), which is our absolute minimum.
5. Let \( f(x) = x e^{-3x} \).

(a) 3 points Calculate \( f'(x) \)

**Solution:** We use the product rule and the chain rule:

\[
f'(x) = e^{-3x} - 3x e^{-3x}
\]

(b) 3 points Calculate \( f''(x) \)

**Solution:** Taking the derivative of the above gives

\[
f''(x) = -3e^{-3x} - 3e^{-3x} + 9x e^{-3x}
\]

which simplifies to

\[
9x e^{-3x} - 6e^{-3x}
\]

(c) 3 points For what values of \( x \) is \( f(x) \) increasing?

**Solution:** To answer this, we need to know when \( f'(x) > 0 \), that is, where

\[
e^{-3x} - 3x e^{-3x} > 0
\]

Factoring out the exponential term gives \( e^{-3x} (1 - 3x) > 0 \), and since \( e^{-3x} \) is always positive, we only need ask where \( 1 - 3x > 0 \). This happens for

\[
x < \frac{1}{3}.
\]

(d) 3 points For what values of \( x \) is \( f(x) \) concave down?

**Solution:** We need to know when \( f''(x) < 0 \), so factor \( f''(x) \) as

\[
3e^{-3x} (3x - 2).
\]

As before, we can ignore the exponential term, since it is always positive, and we see that \( f''(x) < 0 \) when \( x < 2/3 \).
6. [10 points] A leaky oil tanker is anchored offshore. Because the water is very calm, the oil slick always stays circular as it expands, with a uniform depth of 1 meter. If the oil is leaking from the tanker at a rate of \(100 \text{ m}^3/\text{hr}\), how fast is the radius of the slick expanding (in \(\text{m/ hr}\)) when the diameter is 16 meters?

**Solution:** First, notice that since we are given the rate of oil leaking out from the tanker, this is the rate of change of volume of oil (\(dV/dt = 100 \text{ m}^3/\text{hr}\)), and we want to know the rate of change of the radius (\(dr/dt\)). This means we need to write a formula for the volume of the oil as a function of the radius.

Since we are told that the oil slick is circular and has a constant depth of 1 meter, it is a cylinder of height 1 and radius \(r\). That is,

\[
V = \pi r^2
\]

Taking the derivative with respect to time gives

\[
\frac{dV}{dt} = 2\pi r \frac{dr}{dt}
\]

and plugging in the given values yields

\[
100 = 2\pi \cdot 8 \frac{dr}{dt},
\]

so we have

\[
\frac{dr}{dt} = \frac{100}{16\pi} = \frac{25}{4\pi}.
\]