1. For each of the functions $f(x)$ given below, find $f'(x)$.

(a) $f(x) = \frac{1 + 2x^2}{1 + x^4}$

**Solution:** This is a straightforward quotient rule problem:

$$f'(x) = \frac{(4x)(1 + x^4) - (1 + 2x^2)(4x^3)}{(1 + x^4)^2} = \frac{4x - 4x^3 - 4x^5}{(1 + x^4)^2}$$

The simplification is not required.

(b) $f(x) = \sin(2x) \cos(x)$

**Solution:** Apply the product rule, with a chain rule for the $\sin(2x)$ term to get

$$f'(x) = 2 \cos(2x) \cos(x) - \sin(2x) \sin(x).$$

(c) $f(x) = \arctan(\sqrt{1 + 3x})$

**Solution:** Applying the chain rule, we get

$$\frac{1}{1 + (\sqrt{1 + 3x})^2} \cdot \frac{1}{2} (1 + 3x)^{-1/2} \cdot (3) = \frac{3}{2(2 + 3x)\sqrt{1 + 3x}}$$

(d) $f(x) = \ln(\tan(x))$

**Solution:** Another chain rule problem:

$$f'(x) = \frac{1}{\tan(x)} \cdot \sec^2(x) = \frac{\cos(x)}{\sin(x) \cos^2(x)} = \sec(x) \csc(x).$$

2. Compute each of the following derivatives as indicated:

(a) $\frac{d}{dt} \left[ e^{\sin^2(t)} \right]$

**Solution:** The chain rule gives

$$e^{\sin^2(t)} \cdot 2 \sin(t) \cdot (\cos(t)) = -2 \sin(t) \cos(t) e^{\sin^2(t)}$$
(b) \( \frac{d}{du} \left[ u^5 \ln(\sin(u)) \right] \)

**Solution:** Using the product rule (and the chain rule), we obtain

\[
5u^4 \ln(\sin(u)) + u^5 \frac{1}{\sin(u)} \cos(u) = u^4 \left( 5 \ln(\sin(u)) + u \cot(u) \right)
\]

(c) \( \frac{d}{dz} \left[ \sqrt{1 + \sqrt{1 + z}} \right] \)

**Solution:** View this as \( \frac{d}{dz} \left[ \left( 1 + (1 + z)^{1/2} \right)^{1/2} \right] \) and apply the chain rule:

\[
\frac{1}{2} \left( 1 + (1 + z)^{1/2} \right)^{-1/2} \cdot \frac{1}{2} (1 + z)^{-1/2} = \frac{1}{4\sqrt{1+z} \sqrt{1+\sqrt{1+z}}}
\]

(d) \( \frac{d}{dx} \left[ e^x - \pi^2 \right] \)

**Solution:** Remembering that \( \pi^2 \) is a constant, the derivative is just \( e^x \).

3. The curve \( x^2 - xy + y^2 = 16 \) is an ellipse centered at the origin.

(a) Find the points where this ellipse intersects the \( x \)-axis.

**Solution:** Since we are looking for points on the \( x \)-axis, this is where \( y = 0 \). Substituting \( y = 0 \) into the equation of the ellipse gives

\[ x^2 = 16 \quad \text{so} \quad x = \pm 4. \]

(b) Find the slope of the tangent line to this ellipse at each of the points from part (a).

**Solution:** Using implicit differentiation, we obtain

\[
2x - \left( y + x \frac{dy}{dx} \right) + 2y \frac{dy}{dx} = 0.
\]

Substituting \( y = 0 \) and \( x = \pm 4 \) yields

\[ \pm 8 = \pm 4 \frac{dy}{dx} \]

and so the slope at either point is 2.
(c) Locate all points on this ellipse where the line tangent to the curve is horizontal.

**Solution:** To do this, we need to find all points \((x, y)\) where the slope of the tangent line is zero. From part (b), we have

\[
2x - \left(y + x \frac{dy}{dx}\right) + 2y \frac{dy}{dx} = 0;
\]

solving this for \(dy/dx\) gives

\[
\frac{dy}{dx} = \frac{y - 2x}{2y - x}.
\]

Thus, the slope of the tangent line will be zero when \(y = 2x\).

Now we go back to the equation of the ellipse \((x^2 - xy + y^2 = 16)\) and substitute \(y = 2x\) to obtain

\[
x^2 - x(2x) + (2x)^2 = 16, \quad \text{or equivalently,} \quad 3x^2 = 16.
\]

Thus, \(x = \pm 4/\sqrt{3}\). Since \(y = 2x\), we have \(y = \pm 8/\sqrt{3}\). Thus, the two points in question are

\[
\left(\frac{4}{\sqrt{3}}, \frac{8}{\sqrt{3}}\right) \quad \text{and} \quad \left(-\frac{4}{\sqrt{3}}, -\frac{8}{\sqrt{3}}\right)
\]

4. Let \(f(x) = x \ln(2x)\)

(a) Calculate \(f'(x)\)

**Solution:** Applying the product rule (and the chain rule) gives

\[
f'(x) = \ln(2x) + x \cdot \frac{1}{2x} \cdot 2 = \ln(2x) + 1.
\]

(b) Calculate \(f''(x)\)

**Solution:** Taking the derivative of the above, we get \(f''(x) = \frac{1}{x}\).

(c) For what values of \(x\) is \(f(x)\) increasing?

**Solution:** As we all know, \(f(x)\) is increasing when \(f'(x) > 0\). Thus, using our answer from part (a) tells us that we need to know when

\[
\ln(2x) + 1 > 0 \quad \text{or, equivalently,} \quad \ln(2x) > -1.
\]

Exponentiating both sides gives \(2x > e^{-1}\), so we know that

\[
f(x) \text{ is increasing for } x > \frac{1}{2e}.
\]
(d) For what values of $x$ is $f(x)$ concave down?

**Solution:** We need to determine when $f''(x) < 0$. From part (b), this means

$$\frac{1}{x} < 0$$

that is, $x < 0$.

However, remember that $\ln(3x)$ is only defined for $x > 0$. Thus $f(x)$ is concave up for all values of $x$ in its domain. There are no values of $x$ where $f(x)$ is concave down.

5. The volume $V$ of a spherical ball is growing at a constant rate of $1 \text{ m}^3/\text{min}$. Determine the rate of increase of its surface area $S$ (in $\text{m}^2/\text{min}$) when its radius $r$ is equal to 1 meter.

Perhaps you might find it helpful to recall that the volume of a sphere of radius $r$ is given by $V = \frac{4}{3}\pi r^3$, and its surface area is $S = 4\pi r^2$.

**Solution:** The statement that the volume is growing at $1 \text{ m}^3/\text{min}$, we have $\frac{dV}{dt} = 1$. We are asked to find the rate of increase of the surface area when the radius is 1, that is, $\frac{dS}{dt}$ when $r = 1$.

We know that

$$V = \frac{4}{3}\pi r^3$$

so

$$\frac{dV}{dt} = 4\pi r \frac{dr}{dt}$$

When $r = 1$, the equation on the right gives us $1 = 4\pi (1) \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{4\pi}$.

Now we use

$$S = 4\pi r^2$$

to get

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$ 

Since $r = 1$ and $\frac{dr}{dt} = \frac{1}{4\pi}$, we have

$$\frac{dS}{dt} = 8\pi \frac{1}{4\pi} = 2.$$
6. Use a linear approximation to estimate the value of \( \arcsin(0.52) \)

**Solution:** We use the following two facts:

- \( f(x) \approx f(a) + f'(a)(x - a) \) for \( x \) near \( a \),
- \( \arcsin(0.5) = \pi/6 \).

Thus, if we take \( a = \frac{1}{2} \) and \( f(x) = \arcsin(x) \), we can approximate \( f(0.52) \) using the tangent line.

Recalling that \( f'(a) = \frac{1}{\sqrt{1 - a^2}} \), we have

\[
f'(1/2) = \frac{1}{\sqrt{1 - (1/2)^2}} = \frac{1}{\sqrt{3/4}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}.
\]

Thus, we have

\[
\arcsin(0.52) \approx \frac{\pi}{6} + \frac{2}{\sqrt{3}} (0.52 - 0.5) = \frac{\pi}{6} + 0.04.
\]

If you prefer to phrase this in terms of differentials, you get the same answer. The differential of \( \arcsin(x) \) is \( dy = \frac{dx}{\sqrt{1-x^2}} \). Taking \( x = \frac{1}{2} \) and \( dx = 0.02 \), we have

\[
\arcsin(0.52) \approx \arcsin(1/2) + dy = \frac{\pi}{6} + \frac{0.04}{\sqrt{3}}.
\]

This is approximately \( \frac{\pi}{6} + 0.023094 \) while \( \arcsin(0.52) \) is \( \frac{\pi}{6} + 0.023252 \) to 6 places. Obviously, you wouldn’t have been able to determine that without a calculator.

Note that the function \( \arcsin(x) \) gives a result in radians. If you gave an answer in degrees, I suspect that you got the derivative all wrong…that is, you neglected to adjust by \( 180/\pi \).