1. For each of the functions \( f(x) \) given below, find \( f'(x) \).

   (a) \( f(x) = \frac{1 + 2x^2}{1 + x^3} \)

   **Solution:** This is a straightforward quotient rule problem:
   \[
   f'(x) = \frac{(4x)(1 + x^3) - (1 + 2x^2)(3x^2)}{(1 + x^3)^2} = \frac{4x - 3x^2 - 2x^4}{(1 + x^3)^2}
   \]
   The simplification is not required.

   (b) \( f(x) = \sin(4x) \cos(x) \)

   **Solution:** Apply the product rule, with a chain rule for the \( \sin(4x) \) term to get
   \[
   f'(x) = 4 \cos(4x) \cos(x) - \sin(4x) \sin(x).
   \]

   (c) \( f(x) = \arctan\left(\sqrt{1 + 2x}\right) \)

   **Solution:** Applying the chain rule, we get
   \[
   f'(x) = \frac{1}{1 + (\sqrt{1 + 2x})^2} \cdot \frac{1}{2}(1 + 2x)^{-1/2} \cdot (2) = \frac{1}{(2 + 2x)\sqrt{1 + 2x}}
   \]

   (d) \( f(x) = \ln(\tan(x)) \)

   **Solution:** Another chain rule problem:
   \[
   f'(x) = \frac{1}{\tan(x)} \cdot \sec^2(x) = \frac{\cos(x)}{\sin(x)\cos^2(x)} = \sec(x)\csc(x).
   \]

2. Compute each of the following derivatives as indicated:

   (a) \( \frac{d}{dt} \left[ e^{\sin^2(t)} \right] \)

   **Solution:** The chain rule gives
   \[
   e^{\sin^2(t)} \cdot 2 \sin(t) \cdot (-\cos(t)) = -2 \sin(t) \cos(t) e^{\sin^2(t)}
   \]
4 points (b) \[ \frac{d}{du}\left[u^3 \ln(\sin(u))\right] \]

**Solution:** Using the product rule (and the chain rule), we obtain
\[
3u^2 \ln(\sin(u)) + u^3 \frac{1}{\sin(u)} \cos(u) = u^2 \left(3 \ln(\sin(u)) + u \cot(u)\right)
\]

4 points (c) \[ \frac{d}{dz}\left[\sqrt{1 + \sqrt{1 + z}}\right] \]

**Solution:** View this as \[ \frac{d}{dz}\left[(1 + (1 + z)^{1/2})^{1/2}\right] \] and apply the chain rule:
\[
\frac{1}{2}(1 + (1 + z)^{1/2})^{-\frac{1}{2}} \cdot \frac{1}{2}(1 + z)^{-\frac{1}{2}} = \frac{1}{4\sqrt{1 + z} \sqrt{1 + \sqrt{1 + z}}}
\]

4 points (d) \[ \frac{d}{dx}\left[e^x - \pi^2\right] \]

**Solution:** Remembering that \( \pi^2 \) is a constant, the derivative is just \( e^x \).

3. The curve \( x^2 - xy + y^2 = 4 \) is an ellipse centered at the origin.

4 points (a) Find the points where this ellipse intersects the \( x \)-axis.

**Solution:** Since we are looking for points on the \( x \)-axis, this is where \( y = 0 \). Substituting \( y = 0 \) into the equation of the ellipse gives
\[
x^2 = 4 \quad \text{so} \quad x = \pm 2.
\]

6 points (b) Find the slope of the tangent line to this ellipse at each of the points from part (a).

**Solution:** Using implicit differentiation, we obtain \( 2x - \left(y + x \frac{dy}{dx}\right) + 2y \frac{dy}{dx} = 0 \).

Substituting \( y = 0 \) and \( x = \pm 2 \) yields
\[
\pm 4 = \pm 2 \frac{dy}{dx}
\]
and so the slope at either point is 2.
5 points  
(c) Locate all points on this ellipse where the line tangent to the curve is horizontal.

**Solution:** To do this, we need to find all points \( (x, y) \) where the slope of the tangent line is zero. From part (b), we have

\[
2x - \left( y + x \frac{dy}{dx} \right) + 2y \frac{dy}{dx} = 0;
\]
solving this for \( \frac{dy}{dx} \) gives

\[
\frac{dy}{dx} = \frac{y - 2x}{2y - x}.
\]

Thus, the slope of the tangent line will be zero when \( y = 2x \).

Now we go back to the equation of the ellipse \( x^2 - xy + y^2 = 4 \) and substitute \( y = 2x \) to obtain

\[
x^2 - x(2x) + (2x)^2 = 4, \quad \text{or equivalently,} \quad 3x^2 = 4.
\]

Thus, \( x = \pm 2/\sqrt{3} \). Since \( y = 2x \), we have \( y = \pm 4/\sqrt{3} \). Thus, the two points in question are

\[
\left( \frac{2}{\sqrt{3}}, \frac{4}{\sqrt{3}} \right) \quad \text{and} \quad \left( -\frac{2}{\sqrt{3}}, -\frac{4}{\sqrt{3}} \right)
\]

4. Let \( f(x) = x \ln(4x) \)

4 points  
(a) Calculate \( f'(x) \)

**Solution:** Applying the product rule (and the chain rule) gives

\[
f'(x) = \ln(4x) + x \frac{1}{4x} \cdot 4 = \ln(4x) + 1.
\]

4 points  
(b) Calculate \( f''(x) \)

**Solution:** Taking the derivative of the above, we get \( f''(x) = \frac{1}{x} \).

3 points  
(c) For what values of \( x \) is \( f(x) \) increasing?

**Solution:** As we all know, \( f(x) \) is increasing when \( f'(x) > 0 \). Thus, using our answer from part (a) tells us that we need to know when

\[
\ln(4x) + 1 > 0 \quad \text{or, equivalently,} \quad \ln(4x) > -1.
\]

Exponentiating both sides gives \( 4x > e^{-1} \), so we know that \( f(x) \) is increasing for \( x > \frac{1}{4e} \).
(d) For what values of \( x \) is \( f(x) \) concave down?

**Solution:** We need to determine when \( f''(x) < 0 \). From part (b), this means

\[
\frac{1}{x} < 0 \quad \text{that is,} \quad x < 0.
\]

However, remember that \( \ln(2x) \) is only defined for \( x > 0 \). Thus \( f(x) \) is concave up for all values of \( x \) in its domain. There are no values of \( x \) where \( f(x) \) is concave down.

5. The volume \( V \) of a spherical ball is growing at a constant rate of \( 1 \ m^3/\text{min} \). Determine the rate of increase of its surface area \( S \) (in \( m^2/\text{min} \)) when its radius \( r \) is equal to 1 meter.

Perhaps you might find it helpful to recall that the volume of a sphere of radius \( r \) is given by \( V = \frac{4}{3} \pi r^3 \), and its surface area is \( S = 4\pi r^2 \).

**Solution:** The statement that the volume is growing at \( 1 \ m^3/\text{min} \), we have \( \frac{dV}{dt} = 1 \). We are asked to find the rate of increase of the surface area when the radius is 1, that is, \( \frac{dS}{dt} \) when \( r = 1 \).

We know that

\[
V = \frac{4}{3} \pi r^3 \quad \text{so} \quad \frac{dV}{dt} = 4\pi r \frac{dr}{dt}
\]

When \( r = 1 \), the equation on the right gives us \( 1 = 4\pi (1) \frac{dr}{dt} \), so \( \frac{dr}{dt} = \frac{1}{4\pi} \).

Now we use

\[
S = 4\pi r^2 \quad \text{to get} \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}.
\]

Since \( r = 1 \) and \( \frac{dr}{dt} = \frac{1}{4\pi} \), we have

\[
\frac{dS}{dt} = 8\pi \frac{1}{4\pi} = 2.
\]
6. Use a linear approximation to estimate the value of \(\arcsin(0.51)\)

**Solution:** We use the following two facts:

- \(f(x) \approx f(a) + f'(a)(x - a)\) for \(x\) near \(a\),
- \(\arcsin(0.5) = \pi/6\).

Thus, if we take \(a = 1/2\) and \(f(x) = \arcsin(x)\), we can approximate \(f(0.51)\) using the tangent line.

Recalling that \(f'(a) = \frac{1}{\sqrt{1 - a^2}}\), we have

\[
f'(1/2) = \frac{1}{\sqrt{1 - (1/2)^2}} = \frac{1}{\sqrt{3}/2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}.
\]

Thus, we have

\[
\arcsin(0.51) \approx \frac{\pi}{6} + \frac{2}{\sqrt{3}}(0.51 - 0.5) = \frac{\pi}{6} + \frac{0.02}{\sqrt{3}}.
\]

If you prefer to phrase this in terms of differentials, you get the same answer. The differential of \(\arcsin(x)\) is \(dy = \frac{dx}{\sqrt{1 - x^2}}\). Taking \(x = 1/2\) and \(dx = 0.01\), we have

\[
\arcsin(0.51) \approx \arcsin(1/2) + dy = \frac{\pi}{6} + \frac{0.02}{\sqrt{3}}.
\]

This is approximately \(\frac{\pi}{6} + 0.011547\) while \(\arcsin(0.51)\) is \(\frac{\pi}{6} + 0.011586\) to 6 places. Obviously, you wouldn’t have been able to determine that without a calculator.

Note that the function \(\arcsin(x)\) gives a result in radians. If you gave an answer in degrees, I suspect that you got the derivative all wrong…that is, you neglected to adjust by \(180/\pi\).