Math 125 Solutions to First Midterm

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

(a) $\lim_{x \to 3} \frac{x^2 - 9}{6x(x - 3)}$

Solution:

\[
\lim_{x \to 3} \frac{(x - 3)(x + 3)}{6x(x - 3)} = \lim_{x \to 3} \frac{x + 3}{6x} = \frac{3 + 3}{18} = \frac{1}{3}.
\]

(b) $\lim_{x \to \infty} 6 \cos \left(\frac{\pi}{x}\right)$

Solution:

\[
\lim_{x \to \infty} 6 \cos\left(\frac{\pi}{x}\right) = 6 \cos(0) = 6.
\]

(c) $\lim_{x \to 2} \frac{x^2}{(x - 2)^3}$

Solution: Note for $x$ close to 2, the numerator is close to 4 while the denominator tends towards zero. Thus, the function becomes unbounded at 2. Note also that the denominator is always positive. Hence, the limit is $+\infty$.

2. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

(a) $\lim_{x \to \infty} \frac{x^2 - 1}{6x(x - 1)}$

Solution: For $x$ very large, $x^2 - 1 \approx x^2$, and $x - 1 \approx x$. Thus

\[
\lim_{x \to \infty} \frac{x^2 - 1}{6x(x - 1)} = \lim_{x \to \infty} \frac{x^2}{6x(x)} = \lim_{x \to \infty} \frac{1}{6} = \frac{1}{6}.
\]

(b) $\lim_{h \to 3} \frac{(x + h)^2 - x^2}{h}$

Solution:

\[
\lim_{h \to 3} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 3} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 3} \frac{2xh + h^2}{h} = \lim_{h \to 3} 2x + h = 2x + 3.
\]
3 points (c) \( \lim_{x \to -\infty} e^x \cos(x) \)

**Solution:** Observe that for any \( x \), we have \(-1 \leq \cos(x) \leq 1\), and so we also have \(-e^x \leq e^x \cos(x) \leq e^x\). Applying the squeeze theorem,

\[
\lim_{x \to -\infty} -e^x \leq \lim_{x \to -\infty} e^x \cos(x) \leq \lim_{x \to -\infty} e^x,
\]

that is,

\[
0 \leq \lim_{x \to -\infty} e^x \cos(x) \leq 0.
\]
Hence, the limit is 0.

3. Let \( f(x) = 2x^3 - 5x + 2 \).

3 points (a) Find the slope of the secant line passing through the points on the curve \( y = f(x) \) where \( x = 0 \) and \( x = 1 \).

**Solution:** The slope of a line is the ratio of the change in \( y \) to the change in \( x \). Here we have

\[
\text{slope} = \frac{f(1) - f(0)}{1 - 0} = \frac{-1 - 2}{1} = -3.
\]

3 points (b) Find \( f'(1) \).

**Solution:** Using the power rule, \( f'(x) = 6x^2 - 5 \), so \( f'(1) = 1 \).

3 points (c) Write the equation of the tangent line to the graph of \( y = f(x) \) when \( x = 1 \).

**Solution:** The point \((1, f(1))\) is on both the curve and the line. Now, \( f(1) = 2 - 5 = -3 \). We just need the equation of the line of slope 1 passing through the point \((1, -3)\). This is

\[
y + 3 = (x - 1) \quad \text{or} \quad y = x - 4.
\]

3 points (d) At \( x = 1 \), is \( f(x) \) concave up, concave down, or neither? Justify your answer fully.

**Solution:** Since \( f''(x) = 12x \), we know \( f''(1) > 0 \). Thus \( f(x) \) is concave up at \( x = 1 \).
4. For what values of $x$ is the function \( f(x) = \frac{e^x}{4 - e^{1/x}} \) continuous?

**Solution:** Since \( f(x) \) is a composition of exponentials and rational functions, it is continuous everywhere on its domain.

Since \( 1/x \) is not defined for \( x = 0 \), the function is not continuous there.

Furthermore, there will be a discontinuity when the denominator is zero. That is, where \( 4 - e^{1/x} = 0 \), or

\[
4 = e^{1/x}
\]

\[
\ln(4) = \ln \left( e^{1/x} \right) = 1/x
\]

\[
x = \frac{1}{\ln(4)}.
\]

Thus, \( f(x) \) is continuous at all real numbers except \( x = 0 \) and \( x = \frac{1}{\ln(4)} \).

5. Write a limit that represents the slope of the graph

\[ y = \begin{cases} 
|x|^x & x \neq 0 \\
1 & x = 0
\end{cases} \]

at \( x = 0 \). You **do not need to evaluate the limit**.

**Solution:** We just use the definition of the derivative at \( x = 0 \):

\[
f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}.
\]

Since \( h \) is not zero, \( f(h) = |h|^h \) and \( f(0) = 1 \). So,

\[
f'(0) = \lim_{h \to 0} \frac{|h|^h - 1}{h}.
\]

If you prefer to use the version of the definition \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \), you get the same answer except with \( x \) instead of \( h \).
6. At right is the graph of the derivative $f'(x)$ of a function $f(x)$. Use it to answer each of the following questions.

(a) Is $f(x)$ concave up, concave down, or neither at $x = 0$?

**Solution:** Since the derivative is decreasing at $x = 0$, we know $f(x)$ is concave down there.

(b) Which of the following best represents the graph of $f(x)$? (circle your answer).

**Solution:** The graph of $f(x)$ is

(c) Which of the following best represents the graph of $f''(x)$? (circle your answer).

**Solution:** The graph of $f''(x)$ is
7. Let \( f(x) = \frac{x^2 - 4x}{2(x^2 - 16)} \)

4 points (a) Identify the horizontal asymptotes of \( f(x) \). If there are none, write “NONE”.

**Solution:** A function \( f(x) \) has a horizontal asymptote at \( y = L \) when \( \lim_{x \to \infty} f(x) = L \). So, we have

\[
\lim_{x \to \infty} \frac{x^2 - 4x}{2(x^2 - 16)} = \lim_{x \to \infty} \frac{x^2}{2x^2} = \frac{1}{2}.
\]

Thus, there is a horizontal asymptote \( y = \frac{1}{2} \).

4 points (b) Identify the vertical asymptotes of \( f(x) \). If there are none, write “NONE”.

**Solution:** We have a vertical asymptote at \( x = a \) whenever \( \lim_{x \to a^\pm} f(x) = \pm \infty \). The denominator of \( f(x) \) factors as \( 2(x - 4)(x + 4) \), so we have to look at \( a = 4 \) and \( a = -4 \).

Note that if \( x \neq 4 \) \( x \neq -4 \), we have

\[
f(x) = \frac{x^2 - 4x}{2(x^2 - 16)} = \frac{x(x - 4)}{2(x - 4)(x + 4)} = \frac{x}{2(x + 4)}.
\]

Near \( x = -4 \), we have

\[
\lim_{x \to -4^+} f(x) = +\infty \quad \text{and} \quad \lim_{x \to -4^-} f(x) = -\infty,
\]

so there is a vertical asymptote at \( x = -4 \).

Near \( x = 4 \), we have

\[
\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x}{2(x + 4)} = \frac{4}{2(4 + 4)} = \frac{1}{4}.
\]

Thus, \( x = 4 \) is not a vertical asymptote.

8 points 8. Write a function which expresses the area of a rectangle with a perimeter of 16 feet in terms of its width.

**Solution:** Let’s let \( W \) denote the width of the rectangle (in feet), and \( L \) denote its length. Since the perimeter is 16, we know that

\[
2L + 2W = 16,
\]

or equivalently, \( L = 8 - W \).

Since the area of the rectangle is \( LW \), we have

\[
A(W) = (8 - W)W
\]