1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

(a) $3$ points \( \lim_{x \to 0} \frac{\sin x}{\tan x} \)

**Solution:**
\[
\lim_{x \to 0} \frac{\sin x}{\tan x} = \lim_{x \to 0} \frac{\sin x}{\sin x} \cdot \frac{\sin x}{\cos x} = \lim_{x \to 0} \cos x = \cos(0) = 1
\]

(b) $3$ points \( \lim_{x \to +\infty} \frac{3x^2 - 2x - 1}{x^2 - 1} \)

**Solution:**
\[
\lim_{x \to +\infty} \frac{3x^2 - 2x - 1}{x^2 - 1} = \lim_{x \to +\infty} \frac{3x^2}{x^2} = \lim_{x \to +\infty} 3 = 3.
\]

(c) $3$ points \( \lim_{x \to +\infty} \sqrt{9x^2 + x - 3x} \)

**Solution:**
\[
\lim_{x \to +\infty} \sqrt{9x^2 + x - 3x} = \lim_{x \to +\infty} \left( \sqrt{9x^2 + x - 3x} \right) \frac{\sqrt{9x^2 + x + 3x}}{\sqrt{9x^2 + x + 3x}}
\]
\[
= \lim_{x \to +\infty} \frac{9x^2 + x - 9x^2}{\sqrt{9x^2 + x + 3x}}
\]
\[
= \lim_{x \to +\infty} \frac{x}{\sqrt{9x^2 + x + 3x}}
\]
\[
= \lim_{x \to +\infty} \frac{1}{x} \cdot \frac{x}{\sqrt{9x^2 + x + 3x}}
\]
\[
= \lim_{x \to +\infty} \frac{1}{\sqrt{9 + \frac{1}{x} + 3}} = \frac{1}{\sqrt{9 + 3}} = \frac{1}{6}
\]

(d) $3$ points \( \lim_{x \to 0^-} \frac{1}{x^7} \)

**Solution:** We are only considering $x < 0$, so $1/x^7$ is always negative. As $x$ approaches 0 from the left, $1/x^7$ gets larger and larger in absolute value. Hence \( \lim_{x \to 0^-} \frac{1}{x^7} = -\infty \)
(e) 3 points \[ \lim_{x \to 0} \frac{(4 + x)^2 - 16}{x} \]

**Solution:**

\[ \lim_{x \to 0} \frac{(4 + x)^2 - 16}{x} = \lim_{x \to 0} \frac{16 + 8x + x^2 - 16}{x} = \lim_{x \to 0} \frac{8x + x^2}{x} = \lim_{x \to 0} 8 + x = 8 \]

You could also do this by noticing that this is the definition of \( f'(4) \) where \( f(x) = x^2 \) and use the power rule to see that \( f'(x) = 2x \), so \( f'(4) = 8 \), but I doubt anyone did that.

2. 6 points Let

\[ f(x) = \begin{cases} -3x^2 & \text{if } x < -1, \\ 3\tan\left(\frac{x}{4}\right) & \text{if } -1 \leq x \leq 1, \\ -3x^3 & \text{if } x > 1. \end{cases} \]

For which values of \( x \) is \( f(x) \) continuous? Justify your answer.

**Solution:** At right is the graph of \( f(x) \).

Since \(-3x^2\), \(3\tan\left(\frac{x}{4}\right)\), and \(-3x^3\) are all continuous on their respective domains, we only need to check whether they match up at \(-1\) and \(1\). That is, we need to see whether

\[ \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \quad \text{or} \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x). \]

At \( x = -1 \), we see that \(-3(-1)^2 = -3\) and \(3\tan\left(-\frac{\pi}{4}\right) = -3\), so \( f \) is continuous at \(-1\). But at \( x = 1 \), we have \(-3(1)^3 = -3\) and \(3\tan\left(\frac{\pi}{4}\right) = 3\), so \( f \) is not continuous at \(1\). Thus, \( f \) is continuous for all real numbers except \( x = 1 \).

3. Let \( f(x) = 3x^3 - 5x + 4 \).

(a) 5 points Find \( f'(1) \).

**Solution:** Using the power rule, \( f'(x) = 9x^2 - 5 \), so \( f'(1) = 4 \).

(b) 5 points Write the equation of the line tangent to \( f(x) \) at the point \( P = (1,2) \).

**Solution:** We just need the equation of the line of slope 4 passing through the point \((1,2)\). This is

\[ y - 2 = 4(x - 1) \quad \text{or} \quad y = 4x - 2 \]
4. **6 points** Write a limit that represents the slope of the graph

\[ y = \begin{cases} 
6 + x \ln |x| & x \neq 0 \\
6 & x = 0 
\end{cases} \]

at \( x = 0 \). You **do not need to evaluate the limit.**

**Solution:** To do this, we need to remember the definition of the derivative, which is

\[ f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} . \]

In the current case, \( a = 0 \), so \( f(a + h) = f(h) \). Notice that \( f(0) = 6 \), so we have

\[ \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{(6 + h \ln |h|) - 6}{h} \]

This simplifies to

\[ \lim_{h \to 0} \frac{h \ln |h|}{h} = \lim_{h \to 0} \ln |h| = -\infty , \]

although it wasn’t required for you to do this.

5. At right is the graph of **the derivative** \( f' \) of a function.

(a) **4 points** List all values of \( x \) with \(-3 \leq x \leq 4\) where \( f(x) \) has a local minimum.

**Solution:** A local minimum for \( f(x) \) will occur where \( f'(x) \) changes from negative to positive. This happens at \( x = 3 \).

(b) **4 points** At \( x = -1 \), is \( f(x) \) concave up, concave down, or neither?

**Solution:** We know that a function is concave up when its second derivative is positive, and concave down when \( f'' \) is negative. The graph shows \( f'(x) \), which is decreasing near \( x = -1 \). That means the derivative of \( f'(x) \) is negative near \( x = -1 \), so \( f''(-1) < 0 \). Hence \( f(x) \) is concave down at \( x = -1 \).
6. **16 points** For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box.

- **F**
- **H**
- **G**
- **E**
7. Let \( f(x) = \frac{4 - x^2}{(5 + x)^2} \).

(a) **4 points** Identify the horizontal asymptotes of \( f(x) \). If there are none, write “NONE”.

**Solution:** To find the horizontal asymptotes, we calculate the limit as \( x \to \infty \). Thus,

\[
\lim_{x \to \infty} \frac{4 - x^2}{(5 + x)^2} = \lim_{x \to \infty} \frac{4 - x^2}{25 + 10x + x^2} = \lim_{x \to \infty} \frac{-x^2}{x^2} = -1
\]

So there is a horizontal asymptote at \( y = -1 \).

(b) **4 points** Identify the vertical asymptotes of \( f(x) \). If there are none, write “NONE”.

**Solution:** There will be a vertical asymptote whenever there is a finite value \( x = a \) such that \( f(x) \to \pm \infty \) as \( x \to a \) (this could be a one-sided limit). This happens when the denominator is zero but the numerator is non-zero.

For this function, the denominator is zero when \( x = -5 \), and so we have a vertical asymptote at \( x = -5 \), that is,

\[
\lim_{x \to -5} \frac{4 - x^2}{(5 + x)^2} = \infty.
\]

8. **8 points** An exponential function of the form \( y = Ca^x \) passes through the points \((1,6)\) and \((3,24)\). Find \( C \) and \( a \).

**Solution:** Since the function passes through \((1,6)\) and \((3,24)\), we know that

\[
6 = Ca^1 \quad \text{and} \quad 24 = Ca^3
\]

From the first equation, we know that \( C = 6/a \), and putting this into the second equation, we have \( 24 = (6/a)a^3 \), or \( 4 = a^2 \). Thus \( a = 2 \). (We must have \( a > 0 \), or \( a^x \) doesn’t make sense.)

Since \( a = 2 \), we have \( C = 6/2 = 3 \).

Thus, the function is \( y = 3 \cdot 2^x \).