1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points

(a) $\lim_{x \to 1} \frac{x^2 - 1}{7x(x - 1)}$

**Solution:**

$$\lim_{x \to 1} \frac{(x - 1)(x + 1)}{7x(x - 1)} = \lim_{x \to 1} \frac{(x + 1)}{7x} = \frac{1 + 1}{7} = \frac{2}{7}.$$

3 points

(b) $\lim_{x \to \infty} 4 \cos \left( \frac{\pi}{x} \right)$

**Solution:**

$$\lim_{x \to \infty} 2 \cos \left( \frac{\pi}{x} \right) = 2 \cos(0) = 2.$$

3 points

(c) $\lim_{x \to 1} \frac{x^2}{(x - 1)^2}$

**Solution:** Note for $x$ close to 1, the numerator is close to 1 while the denominator tends towards zero. Thus, the function becomes unbounded at 1. Note also that the denominator is always positive. Hence, the limit is $+\infty$.

2. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points

(a) $\lim_{x \to \infty} \frac{x^2 - 4}{7x(x - 2)}$

**Solution:** For $x$ very large, $x^2 - 4 \approx x^2$, and $x - 2 \approx x$. Thus

$$\lim_{x \to \infty} \frac{x^2 - 4}{7x(x - 2)} = \lim_{x \to \infty} \frac{x^2}{7x(x)} = \lim_{x \to \infty} \frac{1}{7} = \frac{1}{7}.$$

3 points

(b) $\lim_{h \to 1} \frac{(x + h)^2 - x^2}{h}$

**Solution:**

$$\lim_{h \to 1} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 1} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 1} \frac{2xh + h^2}{h} = \lim_{h \to 1} 2x + h = 2x + 1.$$
3 points (c) \( \lim_{x \to -\infty} e^x \cos(x) \)

**Solution:** Observe that for any \( x \), we have \(-1 \leq \cos(x) \leq 1\), and so we also have \(-e^x \leq e^x \cos(x) \leq e^x\). Applying the squeeze theorem,

\[
\lim_{x \to -\infty} (-e^x) \leq \lim_{x \to -\infty} e^x \cos(x) \leq \lim_{x \to -\infty} (e^x),
\]

that is,

\[
0 \leq \lim_{x \to -\infty} e^x \cos(x) \leq 0.
\]

Hence, the limit is 0.

3. Let \( f(x) = 4x^3 - 7x + 2 \).

3 points (a) Find the slope of the secant line passing through the points on the curve \( y = f(x) \) where \( x = 0 \) and \( x = 1 \).

**Solution:** The slope of a line is the ratio of the change in \( y \) to the change in \( x \). Here we have

\[
\text{slope} = \frac{f(1) - f(0)}{1 - 0} = \frac{-1 - 2}{1} = -3.
\]

3 points (b) Find \( f'(1) \).

**Solution:** Using the power rule, \( f'(x) = 12x^2 - 7 \), so \( f'(1) = 5 \).

3 points (c) Write the equation of the tangent line to the graph of \( y = f(x) \) when \( x = 1 \).

**Solution:** The point \((1, f(1))\) is on both the curve and the line. Now, \( f(1) = 4 - 7 = -3 \). We just need the equation of the line of slope 5 passing through the point \((1, -3)\). This is

\[
y + 3 = 5(x - 1) \quad \text{or} \quad y = 5x - 8.
\]

3 points (d) At \( x = 1 \), is \( f(x) \) concave up, concave down, or neither? Justify your answer fully.

**Solution:** Since \( f''(x) = 24x \), we know \( f''(1) > 0 \). Thus \( f(x) \) is concave up at \( x = 1 \).
4. For what values of $x$ is the function $f(x) = \frac{e^x}{3 - e^{1/x}}$ continuous?

**Solution:** Since $f(x)$ is a composition of exponentials and rational functions, it is continuous everywhere on its domain.

Since $1/x$ is not defined for $x = 0$, the function is not continuous there.

Furthermore, there will be a discontinuity when the denominator is zero. That is, where $3 - e^{1/x} = 0$, or

$$3 = e^{1/x}$$

$$\ln(3) = \ln\left(e^{1/x}\right) = 1/x$$

$$x = \frac{1}{\ln(3)}.$$  

Thus, $f(x)$ is continuous at all real numbers except $x = 0$ and $x = \frac{1}{\ln(3)}$.

5. Write a limit that represents the slope of the graph

$$y = \begin{cases} |x|^x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

at $x = 0$. You do not need to evaluate the limit.

**Solution:** We just use the definition of the derivative at $x = 0$:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}.$$  

Since $h$ is not zero, $f(h) = |h|^h$ and $f(0) = 1$. So,

$$f'(0) = \lim_{h \to 0} \frac{|h|^h - 1}{h}.$$  

If you prefer to use the version of the definition $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$, you get the same answer except with $x$ instead of $h$. 

MAT 125: Solutions to First Midterm  
October 13, 2009
6. At right is the graph of the derivative $f'(x)$ of a function $f(x)$. Use it to answer each of the following questions.

(a) Is $f(x)$ concave up, concave down, or neither at $x = 0$?

**Solution:** Since the derivative is decreasing at $x = 0$, we know $f(x)$ is concave down there.

(b) Which of the following best represents the graph of $f(x)$? (circle your answer).

**Solution:** The graph of $f(x)$ is

(c) Which of the following best represents the graph of $f''(x)$? (circle your answer).

**Solution:** The graph of $f''(x)$ is
7. Let \( f(x) = \frac{x^2 - 3x}{4(x^2 - 9)} \)

(a) Identify the horizontal asymptotes of \( f(x) \). If there are none, write “NONE”.

**Solution:** A function \( f(x) \) has a horizontal asymptote at \( y = L \) when \( \lim_{x \to \infty} f(x) = L \). So, we have

\[
\lim_{x \to \infty} \frac{x^2 - 3x}{4(x^2 - 9)} = \lim_{x \to \infty} \frac{x^2}{4x^2} = \frac{1}{4}.
\]

Thus, there is a horizontal asymptote \( y = \frac{1}{4} \).

(b) Identify the vertical asymptotes of \( f(x) \). If there are none, write “NONE”.

**Solution:** We have a vertical asymptote at \( x = a \) whenever \( \lim_{x \to a^\pm} f(x) = \pm \infty \). The denominator of \( f(x) \) factors as \( 4(x - 3)(x + 3) \), so we have to look at \( a = 3 \) and \( a = -3 \).

Note that if \( x \neq 3 \) \( x \neq -3 \), we have

\[
f(x) = \frac{x^2 - 3x}{4(x^2 - 9)} = \frac{x(x - 3)}{4(x-3)(x+3)} = \frac{x}{4(x+3)}
\]

Near \( x = -3 \), we have

\[
\lim_{x \to -3^+} f(x) = +\infty \quad \text{and} \quad \lim_{x \to -3^-} f(x) = -\infty,
\]

so there is a vertical asymptote at \( x = -3 \).

Near \( x = 3 \), we have

\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x}{4(x+3)} = \frac{3}{4(3+3)} = \frac{1}{8}.
\]

Thus, \( x = 3 \) is not a vertical asymptote.

8. Write a function which expresses the area of a rectangle with a perimeter of 12 feet in terms of its width.

**Solution:** Let’s let \( W \) denote the width of the rectangle (in feet), and \( L \) denote its length. Since the perimeter is 12, we know that

\[
2L + 2W = 12,
\]

or equivalently, \( L = 6 - W \).

Since the area of the rectangle is \( LW \), we have

\[
A(W) = (6 - W)W
\]