1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

(a) 3 points $\lim_{x \to 0} \frac{\sin x}{\tan x}$

Solution:

\[
\lim_{x \to 0} \frac{\sin x}{\tan x} = \lim_{x \to 0} \frac{\sin x}{\frac{\sin x}{\cos x}} = \lim_{x \to 0} \cos x = \cos(0) = 1
\]

(b) 3 points $\lim_{x \to +\infty} \frac{5x^2 - 4x - 1}{x^2 - 1}$

Solution:

\[
\lim_{x \to +\infty} \frac{5x^2 - 4x - 1}{x^2 - 1} = \lim_{x \to +\infty} \frac{5x^2}{x^2} = \lim_{x \to +\infty} 5 = 5.
\]

(c) 3 points $\lim_{x \to +\infty} \sqrt{4x^2 + x - 2x}$

Solution:

\[
\lim_{x \to +\infty} \sqrt{4x^2 + x - 2x} = \lim_{x \to +\infty} \left( \sqrt{4x^2 + x - 2x} \right) \frac{\sqrt{4x^2 + x^2 + 2x}}{\sqrt{4x^2 + x^2 + 2x}}
\]
\[
= \lim_{x \to +\infty} \frac{4x^2 + x - 4x^2}{\sqrt{4x^2 + x + 2x}}
\]
\[
= \lim_{x \to +\infty} \frac{x}{\sqrt{4x^2 + x + 2x}}
\]
\[
= \lim_{x \to +\infty} \frac{1}{\sqrt{4 + \frac{1}{x}}} = \frac{1}{\sqrt{4 + 2}} = \frac{1}{4}
\]

(d) 3 points $\lim_{x \to 0^-} \frac{1}{x^5}$

Solution: We are only considering $x < 0$, so $1/x^5$ is always negative. As $x$ approaches 0 from the left, $1/x^5$ gets larger and larger in absolute value. Hence $\lim_{x \to 0^-} \frac{1}{x^5} = -\infty$
(e) 3 points \[ \lim_{x \to 0} \frac{(2 + x)^2 - 4}{x} \]

**Solution:**
\[ \lim_{x \to 0} \frac{(2 + x)^2 - 4}{x} = \lim_{x \to 0} \frac{4 + 4x + x^2 - 4}{x} = \lim_{x \to 0} \frac{4x + x^2}{x} = \lim_{x \to 0} 4 + x = 4 \]

You could also do this by noticing that this is the definition of \( f'(2) \) where \( f(x) = x^2 \) and use the power rule to see that \( f'(x) = 2x \), so \( f'(2) = 4 \), but I doubt anyone did that.

2. 6 points Let \( f(x) = \begin{cases} 3x^2 & \text{if } x < -1, \\ 3 \tan\left(\frac{\pi}{4} x\right) & \text{if } -1 \leq x \leq 1, \\ 3x^3 & \text{if } x > 1. \end{cases} \)

For which values of \( x \) is \( f(x) \) continuous? Justify your answer.

**Solution:** At right is the graph of \( f(x) \).
Since \( 3x^2 \), \( 3 \tan\left(\frac{\pi}{4} x\right) \), and \( 3x^3 \) are all continuous on their respective domains, we only need to check whether they match up at \(-1\) and \(1\). That is, we need to see whether
\[ \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \quad \text{or} \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x). \]

At \( x = -1 \), we see that \( 3(-1)^2 = 3 \) and \( 3 \tan\left(-\frac{\pi}{4}\right) = -3 \), so \( f \) is not continuous at \(-1\). But at \( x = +1 \), we have \( 3(1)^3 = 3 \) and \( 3 \tan\left(\frac{\pi}{4}\right) = 3 \), and so \( f \) is continuous at 1.
Thus, \( f \) is continuous for all real numbers except \( x = -1 \).

3. Let \( f(x) = 2x^3 - 4x + 4 \).
   (a) 5 points Find \( f'(1) \).

**Solution:** Using the power rule, \( f'(x) = 6x^2 - 4 \), so \( f'(1) = 2 \).

(b) 5 points Write the equation of the line tangent to \( f(x) \) at the point \( P = (1,2) \).

**Solution:** We just need the equation of the line of slope 2 passing through the point \((1,2)\). This is
\[ y - 2 = 2(x - 1) \quad \text{or} \quad y = 2x \]
4. **6 points** Write a limit that represents the slope of the graph

\[
y = \begin{cases} 
8 + x \ln|x| & x \neq 0 \\
8 & x = 0 
\end{cases}
\]

at \(x = 0\). You **do not need to evaluate the limit**.

**Solution:** To do this, we need to remember the definition of the derivative, which is

\[
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.
\]

In the current case, \(a = 0\), so \(f(a+h) = f(h)\). Notice that \(f(0) = 8\), so we have

\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{(8 + h \ln|h|) - 8}{h}
\]

This simplifies to

\[
\lim_{h \to 0} \frac{h \ln|h|}{h} = \lim_{h \to 0} \ln|h| = -\infty,
\]

although it wasn’t required for you to do this.

5. At right is the graph of the derivative \(f'\) of a function.

(a) **4 points** List all values of \(x\) with \(-3 \leq x \leq 4\) where \(f(x)\) has a local maximum.

**Solution:** A local maximum for \(f(x)\) will occur where \(f'(x)\) changes from positive to negative. This happens at \(x = 0\).

(b) **4 points** At \(x = -1\), is \(f(x)\) concave up, concave down, or neither?

**Solution:** We know that a function is concave up when its second derivative is positive, and concave down when \(f''\) is negative. The graph shows \(f'(x)\), which is decreasing near \(x = -1\). That means the derivative of \(f''(x)\) is negative near \(x = -1\), so \(f''(-1) < 0\). Hence \(f(x)\) is concave down at \(x = -1\).
6. 16 points For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box.
7. Let \( f(x) = \frac{4 - x^2}{(3 + x)^2} \).

(a) 4 points Identify the horizontal asymptotes of \( f(x) \). If there are none, write “NONE”.

**Solution:** To find the horizontal asymptotes, we calculate the limit as \( x \to \infty \). Thus,
\[
\lim_{x \to \infty} \frac{4 - x^2}{(3 + x)^2} = \lim_{x \to \infty} \frac{4 - x^2}{9 + 6x + x^2} = \lim_{x \to \infty} \frac{-x^2}{x^2} = -1
\]
So there is a horizontal asymptote at \( y = -1 \).

(b) 4 points Identify the vertical asymptotes of \( f(x) \). If there are none, write “NONE”.

**Solution:** There will be a vertical asymptote whenever there is a finite value \( x = a \) such that \( f(x) \to \pm \infty \) as \( x \to a \) (this could be a one-sided limit). This happens when the denominator is zero but the numerator is non-zero.

For this function, the denominator is zero when \( x = -3 \), and so we have a vertical asymptote at \( x = -3 \), that is,
\[
\lim_{x \to -3} \frac{4 - x^2}{(3 + x)^2} = -\infty.
\]

8. 8 points An exponential function of the form \( y = Ca^x \) passes through the points \((1,6)\) and \((3,24)\). Find \( C \) and \( a \).

**Solution:** Since the function passes through \((1,6)\) and \((3,24)\), we know that
\[
6 = Ca^1 \quad \text{and} \quad 24 = Ca^3
\]
From the first equation, we know that \( C = 6/a \), and putting this into the second equation, we have \( 24 = (6/a)a^3 \), or \( 4 = a^2 \). Thus \( a = 2 \). (We must have \( a > 0 \), or \( a^x \) doesn’t make sense.)

Since \( a = 2 \), we have \( C = 6/2 = 3 \).

Thus, the function is \( y = 3 \cdot 2^x \).