Solutions for MAT 125 First Midterm

February 23, 2009

1. Let \( f(x) = x^2 + 3x \) with domain all real numbers. Let \( A = (1, f(1)) \) and \( B = (2, f(2)) \). There is also the point \( C = (x, f(x)) \) with \( x \) close to 1.

(a) Calculate the slope of the line through \( A \) and \( B \).

**Solution.** The line through two points \((x_1, y_1)\) and \((x_2, y_2)\) has slope

\[
m = \frac{y_2 - y_1}{x_2 - x_1}.
\]

In this case take

\[
(x_1, x_2) = A = (1, f(1)) = (1, 4)
\]

and

\[
(x_2, y_2) = B = (2, f(2)) = (2, 10).
\]

This gives

\[
m = \frac{10 - 4}{2 - 1} = 6.
\]

(b) Give an equation for the line through \( A \) and \( B \).

**Solution.** An equation for the line with slope \( m \) which contains a point \((x_1, y_1)\) is

\[
y - y_1 = m(x - x_1).
\]

By part (a) we know that the slope is \( m = 6 \). Taking \((x_1, y_1) = A = (1, 4)\) gives the equation

\[
y - 4 = 6(x - 1)
\]

which can be simplified to

\[
y - 6x + 2 = 0.
\]

(c) Explain that the slope of the line through \( A \) and \( C \) is given by

\[
slope = \frac{x^2 + 3x - 4}{x - 1}.
\]

**Solution.** By the same reasoning used in part (a), the slope of the line through \( A = (1, 4) \) and \( C = (x, f(x)) \) is

\[
slope = \frac{f(x) - 4}{x - 1} = x^2 + 3x - 4x - 1.
\]
(d) Calculate the slope of the tangent line to the graph of $f$ at $A$.

Solution. The slope of the tangent line to the graph of $f$ at $A$ is the limit as $C$ approaches $A$ of the slope of the line through $A$ and $C$. As $C$ approaches $A$, $x$ approaches 1. Using the result of (c), we can write the slope of the tangent line to the graph of $f$ at $A$ as

$$slope = \lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1}.$$ 

To calculate this limit we use the factorization

$$x^2 + 3x - 4 = (x + 4)(x - 1).$$

Now we can calculate the limit:

$$\lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1} = \lim_{x \to 1} \frac{(x + 4)(x - 1)}{x - 1} = \lim_{x \to 1} (x + 4) = 1 + 4 = 5.$$

2.

(a) Calculate the limit

$$\lim_{x \to 2} \frac{3x^2 - 15x + 18}{x - 2}.$$

Solution. Observe that we can factor the numerator as

$$3x^2 - 15x + 18 = 3(x^2 - 5x + 6) = 3(x - 2)(x - 3).$$

This allows us to calculate the limit:

$$\lim_{x \to 2} \frac{3x^2 - 15x + 18}{x - 2} = \lim_{x \to 2} \frac{3(x - 2)(x - 3)}{x - 2} = \lim_{x \to 2} 3(x - 3) = 3(2 - 3) = -1.$$

(b) Calculate the limit

$$\lim_{x \to 2} \frac{3x^2 - 15x + 19}{x - 2}.$$
Solution. This limit does not exist (even as an infinite limit). First note that
\[
\lim_{x \to 2^-} \frac{1}{x - 2} = -\infty, \quad \text{and} \quad \lim_{x \to 2^+} \frac{1}{x - 2} = +\infty.
\]
Since \(\lim_{x \to 2^-} (3x^2 - 15x + 19) = 1\), the limit laws (which are valid for infinite limits) tell us that
\[
\lim_{x \to 2^-} \frac{3x^2 - 15x + 19}{x - 2} = \left( \lim_{x \to 2^-} (3x^2 - 15x + 19) \right) \left( \lim_{x \to 2^-} \frac{1}{x - 2} \right) = -\infty.
\]
The analogous calculation shows that
\[
\lim_{x \to 2^+} \frac{3x^2 - 15x + 19}{x - 2} = +\infty.
\]
Since the left limit is not equal to the right limit, we conclude that the limit does not exist.

3. Explain whether the function
\[
f(x) = \begin{cases} 
  \frac{x^2 - 3x}{x^2 - 9} & x \neq 3 \\
  \frac{21}{x} & x = 3 
\end{cases}
\]
is continuous at \(x = 3\) or not.

Solution. The function is continuous at \(x = 3\) if and only if \(\lim_{x \to 3} f(x) = f(3)\). But
\[
\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - 9} = \lim_{x \to 3} \frac{x(x - 3)}{(x - 3)(x + 3)} = \lim_{x \to 3} \frac{x}{x + 3} = \frac{3}{6} = \frac{1}{2}.
\]
Therefore the value of the limit is different from \(f(3) = 21\), so the function is not continuous at \(x = 3\).

4. Given the function
\[
f(x) = \left[ \frac{1}{1 - x} + \frac{1}{x - 3} \right] + \cos(\pi x),
\]
with domain the numbers between 1 and 3, \(1 < x < 3\).
(a) Calculate $f(2)$.

Solution. Since $\cos(2\pi) = 1$,

$$f(2) = \left[\frac{1}{2 - 1} + \frac{1}{2 - 3}\right] + \cos(2\pi) = [1 - 1] + 1 = 0 + 1 = 1.$$

(b) Is there a solution, a number $x$ between 1 and 3, of $f(x) = 0$?

Solution. Yes. First note that

$$f(5/2) = \left[\frac{1}{5/2 - 1} + \frac{1}{5/2 - 3}\right] + \cos(5\pi/2) = \left[\frac{1}{3/2} + \frac{1}{-1/2}\right] + 0 = \frac{2}{3} - 2 = -\frac{4}{3}.$$

The function $f$ is continuous on the closed interval $[2, 5/2]$ and satisfies $f(2) > 0$, $f(5/2) < 0$. By the intermediate value theorem there exists a number $x \in (2, 5/2)$ with $f(x) = 0$.

5. Calculate

$$\lim_{x \to \infty} \frac{3x^2 + 21}{7x^4 + 31x}.$$

Solution. First write

$$\frac{3x^2 + 21}{7x^4 + 31x} = \frac{3x^2 + 21}{7x^4 + 31x} \cdot \frac{1/x^4}{1/x^4} = \frac{3/x^2 + 21/x^4}{7 + 31/x^3}.$$

Using the limit laws and the fact that

$$\lim_{x \to \infty} \frac{1}{x^n} = 0$$

for any positive integer $n$, we get

$$\lim_{x \to \infty} \frac{3x^2 + 21}{7x^4 + 31x} = \lim_{x \to \infty} \frac{3/x^2 + 21/x^4}{7 + 31/x^3} = \frac{3 \lim_{x \to \infty} (1/x) + 21 \lim_{x \to \infty} (1/x^4)}{7 + 31 \lim_{x \to \infty} (1/x^3)} = \frac{3 \cdot 0 + 21 \cdot 0}{7 + 31 \cdot 0} = 0.$$
(a) Calculate

\[ \lim_{x \to 0^+} e^{-1/x}. \]

*Solution.* If \( x > 0 \) then \(-1/x < 0\), and

\[ \lim_{x \to 0^+} (-1/x) = -\infty. \]

By the law for limits of compositions,

\[ \lim_{x \to 0^+} e^{-1/x} = \lim_{y \to -\infty} e^y = 0. \]

(b) Calculate

\[ \lim_{x \to 0^-} e^{-1/x}. \]

*Solution.* If \( x < 0 \) then \(-1/x > 0\) and

\[ \lim_{x \to 0^-} (-1/x) = +\infty. \]

By the law for limits of compositions,

\[ \lim_{x \to 0^-} e^{-1/x} = \lim_{y \to +\infty} e^y = +\infty. \]

7. Explain in words

\[ \lim_{x \to \infty} f(x) = L. \]

*Solution.* This means that the values of the function \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) sufficiently large.

8. This problem requires a sketch, but I don’t know how to insert one into this file.