

1. For each of the functions  $f(x)$  given below, find  $f'(x)$ .

(a) 3 points  $f(x) = x^9 + 5x^4 + 2x^2 + \pi^2$

**Solution:** Don't forget that  $\pi^2 \approx 9.87$ , so its derivative is 0.

$$f'(x) = 9x^8 + 20x^3 + 4x$$

(b) 3 points  $f(x) = \cos(x) \sin(4x)$

**Solution:** This requires the product rule, and the chain rule.

$$f'(x) = -\sin(x) \sin(4x) + 4 \cos(x) \cos(4x)$$

(c) 3 points  $f(x) = \frac{\sin(x)}{\cos(x)}$

**Solution:** After simplifying  $\frac{\sin(x)}{\cos(x)} = \tan(x)$ , we just remember that the derivative of  $\tan(x)$  is  $\sec^2(x)$ .

Alternatively, if you prefer to use the quotient rule, you should get

$$\frac{\cos(x) \cos(x) + \sin(x) \sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

(d) 3 points  $f(x) = \arctan(x^2)$

**Solution:** This is a straight chain-rule problem.

$$f'(x) = \frac{1}{1 + (x^2)^2} \cdot (2x) = \frac{2x}{1 + x^4}$$

Several people were confused and thought that  $\arctan(x) = \frac{1}{\tan(x)} = \cot(x)$ ; this is nonsense. You should know that  $\arctan(x) = y$  means that  $\tan(y) = x$ .

2. Compute each of the following derivatives as indicated:

(a) 3 points  $\frac{d}{dt} \left[ \frac{e^t - e^{-t}}{e^t + e^{-t}} \right]$

**Solution:** Applying the quotient rule gives us

$$\frac{(e^t + e^{-t})(e^t + e^{-t}) - (e^t - e^{-t})(e^t - e^{-t})}{(e^t + e^{-t})^2} = \frac{(e^{2t} + 2 + e^{-2t}) - (e^{2t} - 2 + e^{-2t})}{(e^t + e^{-t})^2}$$

$$= \frac{4}{(e^t + e^{-t})^2}$$

(b) 3 points  $\frac{d}{du} [ u \ln(u) ]$

**Solution:** Using the product rule, we have

$$1 \cdot \ln(u) + u \cdot \frac{1}{u} = \ln(u) + 1$$

(c) 3 points  $\frac{d}{dz} [ \ln(\sec(3z)) ]$

**Solution:** From the chain rule, we have

$$\frac{1}{\sec(3z)} \cdot \sec(3z) \tan(3z) \cdot 3 = 3 \tan(3z)$$

(d) 3 points  $\frac{d}{dx} [ e^x - x^e ]$

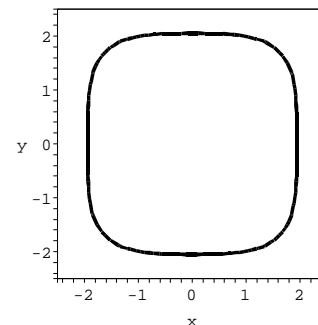
**Solution:** Remembering that  $e$  is a constant, we have  $e^x - e x^{e-1}$ .

3. 10 points Let  $\mathcal{C}$  be the curve which consists of the set of points for which

$$x^4 + x^2 + y^4 = 18$$

(see the graph at right).

Write the equation of the line tangent to  $\mathcal{C}$  which passes through the point  $(1, -2)$ .



**Solution:** In order to write the equation of a line, we need a point on the line (which we have:  $(1, -2)$ ) and the slope of the line. For the slope, we need  $dy/dx$  at the given point.

We *could* solve for  $y$ , getting  $y = \pm \sqrt[4]{18 - x^4 - x^2}$ , and take the derivative of the resulting function to get  $y' = \pm (18 - x^4 - x^2)^{-3/4} (-4x^3 - x)$ .

Instead, let's use implicit differentiation:

$$4x^3 + 2x + 4y^3 \frac{dy}{dx} = 0$$

Since we want the slope when  $x = 1$  and  $y = -2$ , we plug in and solve for  $dy/dx$ .

$$4 + 2 - 32 \frac{dy}{dx} = 0, \quad \text{so} \quad \frac{dy}{dx} = \frac{-6}{-32} = \frac{3}{16}$$

Thus, the desired line is

$$y + 2 = \frac{3}{16}(x - 1) \quad \text{or} \quad y = \frac{3}{16}x - \frac{35}{16}$$

4. 10 points Give the  $x$  and  $y$  coordinates of the (absolute) maximum and minimum values of the function

$$y = x^4 - 8x^2 + 1 \quad \text{where} \quad -1 \leq x \leq 3.$$

**Solution:** First, we locate the critical points. Since the function is a polynomial,  $f'(x)$  is defined everywhere, so we only need concern ourselves with the  $x$  for which  $f'(x) = 0$ .

Since  $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2)$ , we have the critical points

$$x = 0 \quad x = 2 \quad x = -2$$

However, since we are concerned only with  $-1 \leq x \leq 3$ , we discard  $x = -2$ .

Now we evaluate  $f$  at each of the critical points, and the endpoints:

- $f(0) = 1$ .
- $f(2) = 16 - 32 + 1 = -15$ .
- $f(-1) = 1 - 8 + 1 = -6$ .
- $f(3) = 81 - 72 + 1 = 10$ .

The largest value of the above occurs at  $x = 3, y = 10$ . This is our absolute maximum.

The smallest occurs when  $x = 2$  and  $y = -15$ , which is our absolute minimum.

5. Let  $f(x) = x e^{-4x}$ .

(a) 3 points Calculate  $f'(x)$

**Solution:** We use the product rule and the chain rule:

$$f'(x) = e^{-4x} - 4x e^{-4x}$$

(b) 3 points Calculate  $f''(x)$

**Solution:** Taking the derivative of the above gives

$$f''(x) = -4e^{-4x} - 4e^{-4x} + 16x e^{-4x}$$

which simplifies to

$$16x e^{-4x} - 8e^{-4x}$$

(c) 3 points For what values of  $x$  is  $f(x)$  increasing?

**Solution:** To answer this, we need to know when  $f'(x) > 0$ , that is, where

$$e^{-4x} - 4x e^{-4x} > 0$$

Factoring out the exponential term gives  $e^{-4x}(1 - 4x) > 0$ , and since  $e^{-4x}$  is always positive, we only need ask where  $1 - 4x > 0$ . This happens for

$$x < \frac{1}{4}.$$

(d) 3 points For what values of  $x$  is  $f(x)$  concave down?

**Solution:** We need to know when  $f''(x) < 0$ , so factor  $f''(x)$  as

$$8e^{-4x}(2x - 1).$$

As before, we can ignore the exponential term, since it is always positive, and we see that  $f''(x) < 0$  when  $x < 1/2$ .

6. 10 points A leaky oil tanker is anchored offshore. Because the water is very calm, the oil slick always stays circular as it expands, with a uniform depth of 1 meter. If the oil is leaking from the tanker at a rate of  $100 \frac{m^3}{hr}$ , how fast is the radius of the slick expanding (in  $\frac{m}{hr}$ ) when the diameter is 20 meters?

**Solution:** First, notice that since we are given the rate of oil leaking out from the tanker, this is the rate of change of volume of oil ( $dV/dt = 100 \frac{m^3}{hr}$ ), and we want to know the rate of change of the radius ( $dr/dt$ ). This means we need to write a formula for the volume of the oil as a function of the radius.

Since we are told that the oil slick is circular and has a constant depth of 1 meter, it is a cylinder of height 1 and radius  $r$ . That is,

$$V = \pi r^2$$

Taking the derivative with respect to time gives

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt}$$

and plugging in the given values yields

$$100 = 2\pi \cdot 10 \frac{dr}{dt},$$

so we have

$$\frac{dr}{dt} = \frac{100}{20\pi} = \frac{5}{\pi}.$$