MAT 125 Solutions to First Midterm

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

   (a) $\boxed{\text{3 points}}$ $\lim\limits_{x\to 0} \dfrac{\sin x}{\tan x}$

   Solution:
   $$\lim_{x\to 0}\frac{\sin x}{\tan x} = \lim_{x\to 0}\frac{\sin x}{\frac{\sin x}{\cos x}} = \lim_{x\to 0}\cos x = \cos(0) = 1$$

   (b) $\boxed{\text{3 points}}$ $\lim\limits_{x\to +\infty} \dfrac{3x^2-2x-1}{x^2-1}$

   Solution:
   $$\lim_{x\to+\infty}\frac{3x^2-2x-1}{x^2-1} = \lim_{x\to+\infty}\frac{3x^2}{x^2} = \lim_{x\to+\infty} 3 = 3.$$

   (c) $\boxed{\text{3 points}}$ $\lim\limits_{x\to +\infty} \sqrt{9x^2+x}-3x$

   Solution:
   $$\begin{aligned}
   \lim_{x\to+\infty}\sqrt{9x^2+x}-3x &= \lim_{x\to+\infty}\left(\sqrt{9x^2+x}-3x\right)\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}\\
   &= \lim_{x\to+\infty}\frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x}\\
   &= \lim_{x\to+\infty}\frac{x}{\sqrt{9x^2+x}+3x}\\
   &= \lim_{x\to+\infty}\frac{\frac{1}{x}}{\frac{1}{x}}\cdot\frac{x}{\sqrt{9x^2+x}+3x}\\
   &= \lim_{x\to+\infty}\frac{1}{\sqrt{9+\frac{1}{x}}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}
   \end{aligned}$$

   (d) $\boxed{\text{3 points}}$ $\lim\limits_{x\to 0^-} \dfrac{1}{x^7}$

   Solution: We are only considering $x<0$, so $1/x^7$ is always negative. As x approaches 0 from the left, $1/x^7$ gets larger and larger in absolute value. Hence $\lim\limits_{x\to 0^-}\dfrac{1}{x^7} = -\infty$

(e) $\boxed{\text{3 points}}$ $\lim_{x \to 0} \dfrac{(4+x)^2 - 16}{x}$

Solution:

$$\lim_{x \to 0} \frac{(4+x)^2 - 16}{x} = \lim_{x \to 0} \frac{(16+8x+x^2) - 16}{x} = \lim_{x \to 0} \frac{8x+x^2}{x} = \lim_{x \to 0} 8 + x = 8$$

You could also do this by noticing that this is the definition of $f'(4)$ where $f(x) = x^2$ and use the power rule to see that $f'(x) = 2x$, so $f'(4) = 8$, but I doubt anyone did that.

2. $\boxed{\text{6 points}}$ Let

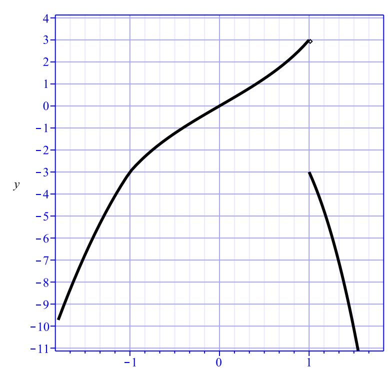
$$f(x) = \begin{cases} -3x^2 & \text{if } x < -1, \\ 3\tan(\frac{\pi}{4}x) & \text{if } -1 \leq x \leq 1, \\ -3x^3 & \text{if } x > 1. \end{cases}$$

For which values of x is $f(x)$ continuous? Justify your answer.

Solution: At right is the graph of $f(x)$.
Since $-3x^2$, $3\tan(\frac{\pi}{4}x)$, and $-3x^3$ are all continuous on their respective domains, we only need to check whether they match up at -1 and 1. That is, we need to see whether

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \quad \text{or} \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x).$$

At $x = -1$, we see that $-3(-1)^2 = -3$ and $3\tan(-\frac{\pi}{4}) = -3$, so f **is** continuous at -1. But at $x = +1$, we have $-3(1)^3 = -3$ and $3\tan(\frac{\pi}{4}) = 3$, so f **is not** continuous at 1. Thus, f is continuous for all real numbers except $x = 1$.



3. Let $f(x) = 3x^3 - 5x + 4$.

(a) $\boxed{\text{5 points}}$ Find $f'(1)$.

Solution: Using the power rule, $f'(x) = 9x^2 - 5$, so $f'(1) = 4$.

(b) $\boxed{\text{5 points}}$ Write the equation of the line tangent to $f(x)$ at the point $P = (1, 2)$.

Solution: We just neet the equation of the line of slope 4 passing through the point $(1, 2)$. This is

$$y - 2 = 4(x-1) \qquad \text{or} \qquad y = 4x - 2$$

4. $\boxed{\text{6 points}}$ Write a limit that represents the slope of the graph

$$y = \begin{cases} 6 + x\ln|x| & x \neq 0 \\ 6 & x = 0 \end{cases}$$

at $x = 0$. You **do not need to evaluate the limit.**

Solution: To do this, we need to remember the definition of the derivative, which is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

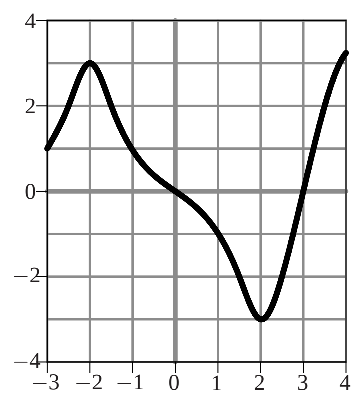
In the current case, $a = 0$, so $f(a+h) = f(h)$. Notice that $f(0) = 6$, so we have

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{(6 + h\ln|h|) - 6}{h}$$

This simplifies to

$$\lim_{h\to 0} \frac{h\ln|h|}{h} = \lim_{h\to 0} \ln|h| = -\infty,$$

although it wasn't required for you to do this.



5. At right is the graph of **the derivative** f' of a function.

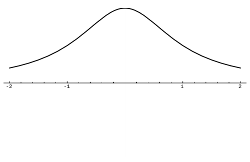
(a) $\boxed{\text{4 points}}$ List all values of x with $-3 \le x \le 4$ where $f(x)$ has a local minimum.

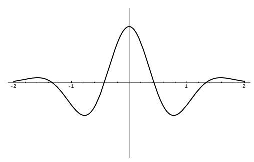
Solution: A local minimum for $f(x)$ will occur where $f'(x)$ changes from negative to positive. This happens at $x = 3$.

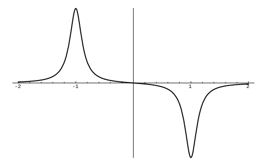
(b) $\boxed{\text{4 points}}$ At $x = -1$, is $f(x)$ concave up, concave down, or neither?

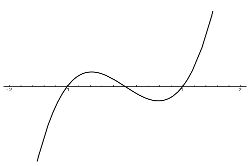
Solution: We know that a function is concave up when its second derivative is positive, and concave down when f'' is negative. The graph shows $f'(x)$, which is decreasing near $x = -1$. That means the derivative of $f'(x)$ is negative near $x = -1$, so $f''(-1) < 0$. Hence $f(x)$ is concave down at $x = -1$.

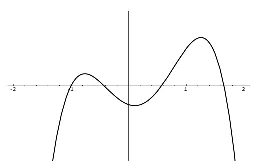
6. $\boxed{\text{16 points}}$ For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box.

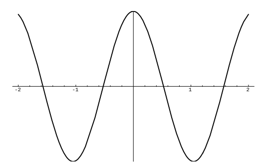
$\boxed{\textbf{F}}$ 

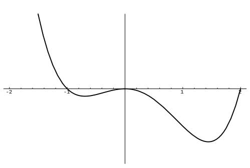
A: 

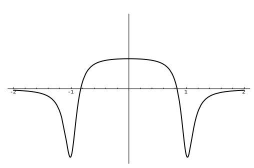
B: 

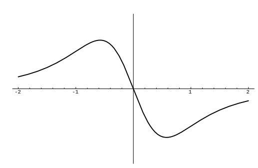
$\boxed{\textbf{H}}$ 

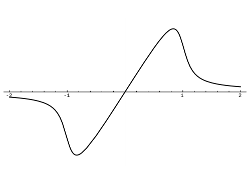
C: 

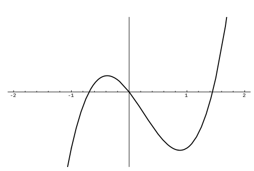
D: 

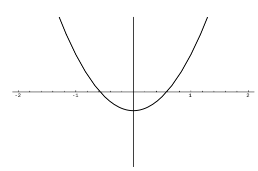
$\boxed{\textbf{G}}$ 

E: 

F: 

$\boxed{\textbf{E}}$ 

G: 

H: 

7. Let $f(x) = \dfrac{4-x^2}{(5+x)^2}$.

   (a) $\boxed{\text{4 points}}$ Identify the horizontal asymptotes of $f(x)$. If there are none, write "NONE".

   Solution: To find the horizontal asymptotes, we calculate the limit as $x \to \infty$. Thus,

   $$\lim_{x\to\infty} \frac{4-x^2}{(5+x)^2} = \lim_{x\to\infty} \frac{4-x^2}{25+10x+x^2} = \lim_{x\to\infty} \frac{-x^2}{x^2} = -1$$

   So there is a horizontal asymptote at $y = -1$.

   (b) $\boxed{\text{4 points}}$ Identify the vertical asymptotes of $f(x)$. If there are none, write "NONE".

   Solution: There will be a vertical asymptote whenever there is a finite value $x = a$ such that $f(x) \to \pm\infty$ as $x \to a$ (this could be a one-sided limit). This happens when the denominator is zero but the numerator is non-zero.

   For this function, the denominator is zero when $x = -5$, and so we have a vertical asymptote at $x = -5$, that is,

   $$\lim_{x\to -5} \frac{4-x^2}{(5+x)^2} = \infty.$$

8. $\boxed{\text{8 points}}$ An exponential function of the form $y = Ca^x$ passes through the points $(1,6)$ and $(3,24)$. Find C and a.

   Solution: Since the function passes through $(1,6)$ and $(3,24)$, we know that

   $$6 = Ca^1 \qquad \text{and} \qquad 24 = Ca^3$$

   From the first equation, we know that $C = 6/a$, and putting this into the second equation, we have $24 = (6/a)a^3$, or $4 = a^2$. Thus $a = 2$. (We must have $a > 0$, or a^x doesn't make sense.)

   Since $a = 2$, we have $C = 6/2 = 3$.

   Thus, the function is $y = 3 \cdot 2^x$