First Midterm Exam- Solutions MAT 125, Fall 2004

1. *(10 points)* A function $T(z)$ is given by the following table:

z	0	1	2	3	4
$T(z)$	5	7	3	-1	-4

a. Write the equation of the line that passes through the points on the graph of $T(z)$ with $z = 1$ and $z = 3$.

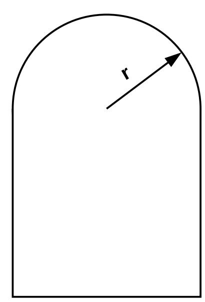
Solution: The slope of the line $m = \frac{T(3)-T(1)}{3-1} = -4$. Applying the point slope formula to the point $(1, T(1))$ yields:

$$y - 7 = -4(z - 1).$$

b. If T is continuous function for $0 \leq z \leq 4$, why can we conclude that $T(z) = 0$ for some value of z between 2 and 3?

Solution: Note that $T(2) > 0$ and $T(3) < 0$. Since T is continuous it does not skip values (I.V.T). That is, $T(z)$ can not skip zero when it passes from the being negative to positive.

2. *(15 points)* A window has the shape of a half-circle on top of a square. Denote the radius of the circle by r. See the figure at right.



a. Express the perimeter P of the window as a function of r.

Solution: The perimeter of the square part is $2r + 2r + 2r$ while the semi-circle contributes $\frac{1}{2}2\pi r$. So

$$P = r(6 + \pi).$$

b. Express the area A of the window as a function of r.

Solution: The area of the semi-circle is $\frac{1}{2}\pi r^2$, while the square yields $(2r)^2$. Hence

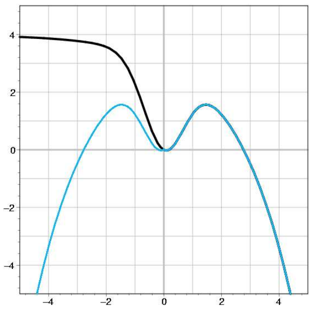
$$A = r^2(\pi/2 + 4).$$

c. Express the area A of the window as a function of its perimeter P.

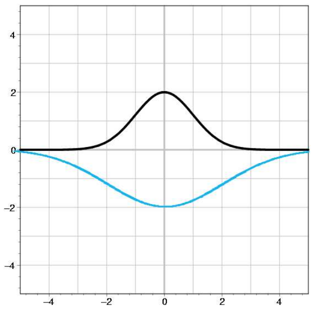
Solution: From [a] we see that $r = \frac{P}{6+\pi}$. Subsituting this equation into [b] yields

$$A = r^2(\pi/2 + 4) = \left(\frac{P}{6 + \pi}\right)^2 (\pi/2 + 4).$$

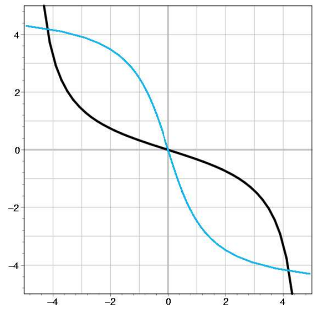
3. *(9 points)* The graphs of several functions $f(x)$ are shown below. On the same set of axes, sketch the function $g(x)$ as indicated.



$g(x) = f(|x|)$



$g(x) = -f(x/2)$



$g(x) = f^{-1}(x)$

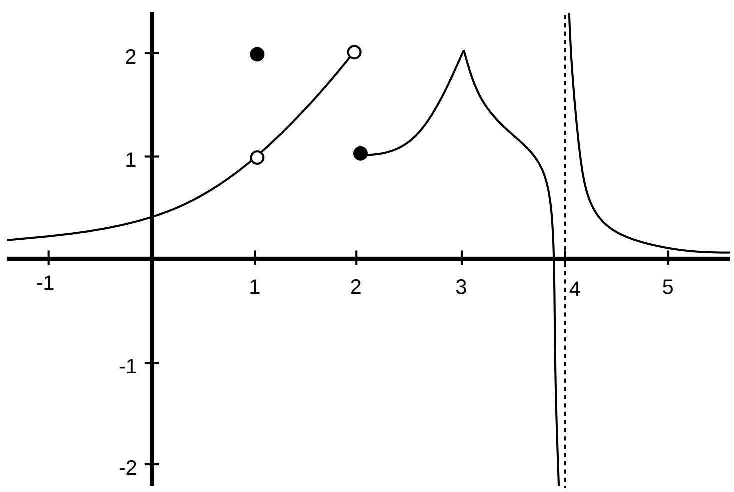
Solution: From left to right:

(1) erase the left half plane part of the graph ($x < 0$) and reflect the right half to the left (over the y-axis). It is the even function constructed by only using non-negative values of x.

(2) Stretch horizontally by two and reflect over the x-axis.

(3) Reflect over the line $y = x$.

4. *(20 points)* Let $f(x)$ be the function whose graph is shown below.



a. List all points $-1 \leq x \leq 5$ where $f(x)$ is not continuous. If there are none, write "none".

Solution: $x = 1, 2, 4$ (breaks or holes in the graph).

b. What is $f(2)$? If it is not defined, write DNE.

Solution: $f(2) = 1$ (we use the closed dot).

c. What is $\lim_{x \to 2} f(x)$? If it is not defined, write DNE.

Solution: The limit DNE because the left and right hand limits are not equal.

d. What is $\lim_{x \to 1^-} f(x)$? If it is not defined, write DNE.

Solution: Both the right and left hand limits are 1.

e. What is $\lim_{x \to 2} \dfrac{f(x-1)}{f(x+1)}$? If it is not defined, write DNE.

Solution: $\lim_{x \to 2} f(x-1) = 1$ and $\lim_{x \to 2} f(x+1) = 2$. Dividing the two limits (the bottom limit is not zero) gives $1/2$.

5. *(20 points)* Compute each of the limits below. If the limit does not exist, say so. Justify your answer in all cases.

a. $\lim_{x \to 1} 2\ln(x)$

Solution:

$$\lim_{x \to 1} 2\ln(x) = 2\ln(1) = 0$$

(This is a continuous function, so just plug in 1).

b. $\lim_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$

Solution: $\lim_{x \to 3} \dfrac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \dfrac{(x-3)(x+2)}{x-3} = \lim_{x \to 3} \dfrac{(x+2)}{1} = 5.$

c. $\lim_{x \to 0^-} \dfrac{|x|}{x}$

Solution:

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1$$

For $x < 0,\ |x| = -x$. So $\frac{|x|}{x} = \frac{-x}{x} = -1$.

d. $\lim_{x \to 0} \dfrac{(x-3)^2 - 9}{x}$

Solution: $\lim_{x \to 0} \dfrac{x^2 - 6x + 9 - 9}{x} = \lim_{x \to 0} x - 6 = -6.$

e. $\lim_{h \to 0^+} \tan(h) \sin\left(\dfrac{\pi}{h}\right)$ Hint: recall that $-1 \le \sin x \le 1$ for any x.

Solution: Use the squeeze theorem. Since $\lim_{h \to 0^+} \tan(h) = 0$ and $\sin\left(\frac{\pi}{h}\right)$ is bounded, the product tends towards zero.

6. *(15 points)*

a. The values of the functions h and g are given by the table at right. What is the value of the function $g \circ h$ at 1?

x	g(x)	h(x)
0	1	2
1	0	1
2	2	0

Solution: Since $h(1) = 1$, $g(h(1)) = g(1) = 0$.

b. If $3^{x+2} = 7$, what is x?

Solution: Taking the $\ln$ of both sides, $(x + 2)\ln(3) = \ln(7)$. Thus $x + 2 = \frac{\ln(7)}{\ln(3)}$ which implies that

$$x = \frac{\ln(7)}{\ln(3)} - 2.$$

c. What is the domain of the function $\ln(x^2 - 1)$?

Solution: We must have $x^2 - 1 > 0$ since the natural log only allows those values. Hence $x^2 > 1$ which implies that $x > 1$, or $x < -1$.

d. If $\sin(x) = 1/3$ and $\tan(x) < 0$, what is $\cos(x)$?

Solution: Since $\tan(x) < 0$ and $\sin(x) > 0$ we must have $\cos(x) < 0$ (Why?). Using the equation $(\sin(x))^2 + (\cos(x))^2 = 1$ we obtain, $\cos(x) = \pm\sqrt{1 - (\sin(x))^2}$. But since $\cos(x) < 0$ we must take the negative sign. Thus

$$\cos(x) = -\sqrt{1 - (1/3)^2} = \frac{2\sqrt{2}}{3}.$$

e. If the graph of $y = e^{kt}$ passes through the point $(5, 1)$, what is k?

Solution: We have that $1 = e^{k \cdot 5}$ which implies that $k = 0$.

7. *(11 points)* Let $g(x) = \dfrac{5-5e^x}{5+5e^x}$.

a. Write a formula for $g^{-1}(x)$.

Solution: Note that the "Fives" all cancel out, so

$$g(x) = \frac{1-e^x}{1+e^x}$$

Interchanging the $x-y$ variables we obtain:

$$x = \frac{1-e^y}{1+e^y}.$$

Now we solve for x. Cross multiplying gives

$$x + xe^y = 1 - e^y.$$

Next factoring gives

$$e^y(x+1) = 1 - x.$$

Hence

$$e^y = \frac{1-x}{1+x}$$

and

$$y = \ln\left(\frac{1-x}{1+x}\right).$$

b. What is the domain of $g^{-1}(x)$?

Solution: We must have[1] $\frac{1-x}{1+x} > 0$. The rational function $\frac{1-x}{1+x}$ can only change sign at $x = 1$ or $x = -1$ (Why?). Simply plugging in and testing three test points: $-5 < -1, 5 > 1, -1 < 0 < 1$, we see that $\frac{1-x}{1+x} > 0$ only when $-1 < x < 1$.

[1]Since this is what is being plugged into the natural log.