This is the solution of homework 7 .

1. $F(y) = (\frac{3}{y^2} - \frac{7}{y^4})(2y - 5y^3)$
   $= \frac{12}{y} - \frac{28}{y^3} + 15y - \frac{35}{y}$
   $= -\frac{23}{y} - \frac{28}{y^3} + 15y$
   Now
   $$\frac{d}{dy}(-\frac{23}{y}) = \frac{23}{y^2}, \quad \frac{d}{dy}(-\frac{28}{y^3}) = \frac{84}{y^2}, \quad \frac{d}{dy}(15y) = 15$$
   Answer: $$\boxed{F'(y) = \frac{23}{y^2} + \frac{84}{y^2} + 15}$$

2. $y = \sqrt{x}(9x^2 - 1)$
   $= x^{\frac{1}{2}}(9x^2 - 1)$
   $= 9x^{\frac{5}{2}} - x^{\frac{1}{2}}.$
   Hence $y' = 9 \times \frac{5}{2}x^{\frac{5}{2}-1} - \frac{1}{2}x^{\frac{1}{2}-1}$
   $= \frac{45}{2}x^{\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$
   Answer: $$\boxed{y' = \frac{45}{2}x^{\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}}$$

3. $y = \frac{7t^2}{5t^2-6t+4}$
   By quotient rule,
   $$y' = \frac{(7t^2)' \cdot (5t^2 - 6t + 4) - 7t^2 \cdot (5t^2 - 6t + 4)'}{(5t^2 - 6t + 4)^2}$$
   Since $(7t^2)' = 14t$ and $(5t^2 - 6t + 4)' = 10t - 6$, we have
   $$y' = \frac{14t \cdot (5t^2 - 6t + 4) - 7t^2 \cdot (10t - 6)}{(5t^2 - 6t + 4)^2}.$$
   Now the numerator is
   $$14t \cdot (5t^2 - 6t + 4) - 7t^2 \cdot (10t - 6) = -42t^2 + 56t = 14t(-3t + 4).$$
   Answer: $$\boxed{y' = \frac{14t(-3t + 4)}{(5t^2 - 6t + 4)^2}.}$$

4. $y = \frac{t^3+5t}{t^4-4}$ By quotient rule,
   $$y' = \frac{(t^3 + 5t)' \cdot (t^4 - 4) - (t^3 + 5t) \cdot (t^4 - 4)'}{(t^4 - 4)^2}$$

Since $(t^3+5t)' = 3t^2+5$ and $(t^4-4)' = 4t^3$, we have

$$y' = \frac{(3t^2+5)\cdot(t^4-4)-(t^3+5t)\cdot(4t^3)}{(t^4-4)^2}.$$

Now the numerator is

$$(3t^2+5)\cdot(t^4-4)-(t^3+5t)\cdot(4t^3) = -t^6-15t^4-12t^2-20.$$

Answer: $$\boxed{y' = \frac{-t^6-15t^4-12t^2-20}{(t^4-4)^2}.}$$

5. $y = \frac{u^6-2u^3+4}{u^2}$
$= u^4 - 2u + \frac{4}{u^2}$
$= u^4 - 2u + 4u^{-2}$

   Since

$$(u^4)' = 4u^3, \quad (2u)' = 2, \quad (4u^{-2})' = -8u^{-3} = -\frac{8}{u^3},$$

   we have $y' = 4u^3 - 2 - \frac{8}{u^3}$

   Answer: $$\boxed{y' = 4u^3 - 2 - \frac{8}{u^3}}$$

6. If $f(2) = 9$, $g(2) = 4$,$f'(2) = -9$, $g'(2) = 3$, find the following numbers.

   a) $(f+g)'(2) = f'(2) + g'(2) = -9 + 3 = -6$

   b) $(fg)'(2) = f'(2)g(2) + f(2)g'(2) = (-9)(4) + (9)(3) = -9$

   c) $(\frac{f}{g})'(2) = \frac{f'(2)g(2)-f(2)g'(2)}{g(2)^2} = \frac{(-9)(4)-(9)(3)}{4^2} = -\frac{63}{16}$

   d)$(\frac{f}{f-g})'(2) = \frac{f'(2)(f(2)-g(2))-f(2)(f'(2)-g'(2))}{(f(2)-g(2))^2} = \frac{(-9)(9-4)-9(-9-3)}{(9-4)^2} = \frac{63}{25}$

7. By product rule, we have $P'(0) = F'(0)G(0) + F(0)G'(0)$. From the graph, we have $F(0) = 6$, $F'(0) = 0$,$G(0) = 4$, $G'(0) = 1$— the slope of the tangent line to the graph of G at $x = 0$ is 1. Hence $P'(0) = 0 \times 4 + 6 \times 1 = 6$

   $Q'(5) = \frac{F'(5)G(5)-F(5)G'(5)}{G(5)^2}$. From the graph, we have $F(5) = 10$, $F'(5) =$ (the slope of the tangent line of F at $x = 5$) $= \frac{1}{2}$,$G(5) = 2$ and $G'(5) =$ (the slope of the tangent line of G at $x = 5$) $= -\frac{4}{3}$. Hence $Q'(5) = \frac{\frac{1}{2}\times 2 - 10\times(-\frac{4}{3})}{2^2} = \frac{43}{12}$

   Answer: $$\boxed{P'(0) = 6, \quad Q'(5) = \frac{43}{12}}$$

8. $f(21) = 11,000$ and $f'(21) = -370$

   Since $R(p) = p \cdot f(p)$, we have, by product rule, $R'(p) = 1 \cdot f(p) + p \cdot f'(p)$ and hence $R'(21) = f(21) + 21 \cdot f'(21) = 11,000 + 21 \times (-370) = 3230$.

   Answer: $\boxed{f'(21) = 3230}$

9. Since f is differentiable everywhere, f is continuous everywhere. In particular, f is continuous at $x = 1$. By definition of continuity, we have

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1-} f(x) = f(1)$$

   . We know $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} x^3 = 1^3 = 1$, $\lim_{x\to 1+} f(x) = \lim_{x\to 1+}(mx + b) = m + b$ and $f(1) = 1$. Therefore we have $m + b = 1$.

   Since f is differentiable everywhere, in particular, f is differentiable at $x = 1$. By definition of derivative, we have

$$\lim_{x\to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1-} \frac{f(x) - f(1)}{x - 1}.$$

   Now

$$\lim_{x\to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1+} \frac{mx + b - (m + b)}{x - 1} = m$$

   since $m + b = 1$.

$$\lim_{x\to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1-} \frac{x^3 - 1}{x - 1} = 3.$$

   Hence $m = 3$. There $b = 1 - m = 1 - 3 = -2$.

   Answer: $\boxed{m = 3, \quad b = -2}$

10. By the given condition, $v(0) = 2m/s$ $s(t) = 2t + 8t^2$ By the definition of velocity, we have $v(t) = s'(t) = 2 + 16t$.

    a) $v(2) = 2 + 16 \times 2 = 34 \quad m/s$

    b) If $v(t) = 2 + 16t = 82$, then $t = 5$.

11. By the given condition, $V(x) = x^3$ and hence $\frac{dV}{dx} = 3x^2$ Hence $\frac{dV}{dx}|_{x=8} = 3 \times 8^2 = 192 \quad \text{mm}^3/\text{mm}$

    Answer: $\boxed{\frac{dV}{dx}|_{x=8} = 192 \quad \text{mm}^3/\text{mm}}$

12. Let t be the time in seconds, r be the radius of the circle in centimeters and A be the area within the circle in cm^2/s. By the given conditions, we have $r = 67t$. We want to find $\frac{dA}{dt}$. Now we have $A = \pi r^2 = \pi(67t)^2$. Hence

$$\frac{dA}{dt} = 2\pi \times 67^2 t.$$

Therefore

a)$\frac{dA}{dt}|_{x=2} = 2\pi \times 67^2 \times 2 = 17956\pi$, $\frac{dA}{dt}|_{x=4} = 2\pi \times 67^2 \times 4 = 35912\pi$ and $\frac{dA}{dt}|_{x=6} = 2\pi \times 67^2 \times 6 = 53868\pi$.

b)The rate of increase of area is constant. (False)
The area is inversely proportional to the elapsed time. (False) Correction: The area is proportional to the elapsed time.
The rate of increase of area grows at a constant rate. (True)
The rate of increase of area is proportional to the square of the elapsed time. (False)
As time goes by, the area grows at an increasing rate.(True).

Answer: $$\boxed{\frac{dA}{dt}|_{x=2} = 17956\pi, \quad \frac{dA}{dt}|_{x=4} = 35912\pi, \quad \frac{dA}{dt}|_{x=6} = 53868\pi.}$$

13. Since $V = 4000(1 - \frac{t}{40})^2$, $0 \le t \le 40$, we have $\frac{dV}{dt} = 4000 \times 2(1 - \frac{t}{40}) \times (-\frac{1}{40}) = 200(-1 + \frac{t}{40}) = 5t - 200$.

Hence a)$V'(5) = -175$gal/min
b) $V'(10) = -150$gal/min
c) $V'(20) = -100$gal/min
d) $V'(40) = 0$gal/min.
e) We need to find the maximal of the absolute value of the $V'(t)$ in the interval $0 \le t \le 40$. Since $V'(t) \le 0$ for all $0 \le t \le 40$. Hence $|V'(t)| = -(5t - 200) = 200 - 5t$. It is easy to see that when $t = 0$, $|V'(0)|$ is the maximal.
f) The same reason as in e), when $t = 40$, $|V'(40)|$ is the minimal, i.e., the water is flowing out the most slowly.

14. Since $PV = nRT$, we have $P'V + PV' = nRT'$. Now $R = 0.0821$, $n = 10$, $P = 7$, $P' = 0.11$, $V = 11$ and $V' = -0.19$. Hence we have

$$T' = \frac{P'V + PV'}{nR} = \frac{0.11 \times 11 + 7 \times (-0.19)}{10 \times 0.0821} \approx -0.1462$$

Answer: $$\boxed{\frac{dT}{dt} \approx -0.1462 \quad \text{K/min}}$$

15. $f(x) = x\sin x$. By product rule, we have $f'(x) = 1 \cdot \sin x + x\cos x = \sin x + x\cos x$.

Answer: $$\boxed{f'(x) = \sin x + x\cos x}$$

16. $y = \frac{1+\text{sin}x}{x+\text{cos}x}$. By quotient rule, we have

$$y' = \frac{(1+\text{sin}x)'(x+\text{cos}x) - (1+\text{sin}x)(x+\text{cos}x)'}{(x+\text{cos}x)^2}$$

$$= \frac{(\text{cos}x)(x+\text{cos}x) - (1+\text{sin}x)(1-\text{sin}x)}{(x+\text{cos}x)^2} = \frac{x\text{cos}x}{(x+\text{cos}x)^2}$$

Answer: $$\boxed{y' = \frac{x\text{cos}x}{(x+\text{cos}x)^2}}$$

17. $y = \text{sec}(\theta)(\text{tan}\theta - \theta)$. By product rule, we have

$$y' = \text{sec}(\theta)'(\text{tan}\theta - \theta) + \text{sec}(\theta)(\text{tan}\theta - \theta)'.$$

Note that $\text{sec}(\theta)' = \text{sec}(\theta)\text{tan}(\theta)$ and $\text{tan}(\theta)' = \text{sec}^2(\theta)$, we have

$$y' = \text{sec}(\theta)\text{tan}(\theta)(\text{tan}\theta - \theta) + \text{sec}(\theta)(\text{sec}^2(\theta) - 1) = \text{sec}(\theta)\text{tan}(\theta)(2\text{tan}\theta - \theta).$$

Answer: $$\boxed{y' = \text{sec}(\theta)\text{tan}(\theta)(2\text{tan}\theta - \theta)}$$

18. Find an equation of the tangent line to the curve $y = \text{cot}(x)$ at the point $(\frac{\pi}{4}, 1)$. To find the the equation of the tangent line, it is enough to find the slope of the line. Note that the slope of the tangent line is exactly the derivative of the function at the point $(\frac{\pi}{4}, 1)$. Now $y' = -\text{csc}^2(x)$ and, in particular, $y'|_{x=\frac{\pi}{4}} = -\text{csc}^2(\frac{\pi}{4}) = -2$.
Therefore the equation of the tangent line at this point is

$$y - 1 = -2(x - \frac{\pi}{4}).$$

Answer: $$\boxed{y - 1 = -2(x - \frac{\pi}{4})}$$

19. a)$f(x) = \sqrt{x}\cos x$.

$$f'(x) = (\sqrt{x})'\cos x + \sqrt{x}(\cos x)' = \frac{1}{2}\frac{1}{\sqrt{x}}\cos x + \sqrt{x}\sin x.$$

b)Note that f is not defined at $x = 0$ and when $x \to 0$, $f(x) \to \infty$. Only the second choice satisfies this condition.

20. $x(t) = 14\cos t$.

a) By definition, $v(t) \equiv x'(t) = -14\sin t$.

b)$x(\frac{5\pi}{6}) = 14\cos\frac{5\pi}{6} = -7\sqrt{3}$.

c)$v(\frac{5\pi}{6}) = -14\sin\frac{5\pi}{6} = -7$. Since the value is negative, it moves to negative x-axis direction.

21. From the given condition in the question, we have

$$\sin\theta = \frac{x}{10}.$$

Hence we have $x = 10\sin\theta$ and $\frac{dx}{d\theta} = 10\cos\theta$. Therefore, $\frac{dx}{d\theta}|_{\theta=\frac{\pi}{3}} = 10\cos\frac{\pi}{3} = 5$.

Answer: $$\boxed{\frac{dx}{d\theta}|_{\theta=\frac{\pi}{3}} = 5 \quad \text{ft/rad}}$$