Solutions to Homework #5 MAT 125

note. I'm using the questions as I see them on the webassign system, your numbers may be different. You should adjust accordingly your solutions and answers

1. $\lim_{x\to\infty} \frac{x^7}{7^x} = 0$

sol: knowing when x becomes large, both x^7 and 7^x becomes large but 7^x goes to ∞ much faster than x^7 the limit is $\underline{\underline{0}}$

In general, $\lim_{x\to\infty} \frac{x^n}{a^x} = 0$ for any $n, a > 1$

2. $\lim_{x\to -9^+} \frac{x+8}{x+9} = -\infty$

sol. as x approaches -9 the numerator approaches -1. the denominator $x+9$ approaches 0. We are approaching -9 from the right, so the denominator is greater than 0. so $\frac{x+8}{x+9}$ becomes a large negative number $\underline{\underline{-\infty}}$

3. $\lim_{x\to\infty} \frac{x^3+3x}{10x^3+5x^2+3} = \frac{1}{10} = 0.1$

sol. dividing both the numerator and the denominator by x^3 we have $\lim_{x\to\infty} \frac{x^3/x^3 + 3x/x^3}{10x^3/x^3 + 5x^2/x^3 + 3/x^3} = \lim_{x\to\infty} \frac{1+3/x^2}{10+5/x+3/x^3}$

as x becomes big $3/x^2$, $5/x$, $3/x^2$ becomes tiny (approaches 0)

so $\lim_{x\to\infty} \frac{x^3+3x}{10x^3+5x^2+3} = \frac{1}{10}$.

4. $\lim_{x \to -\infty} \dfrac{x^2+3x}{7x^3+7x^2+3} = 0$

sol: dividing both the numerator and the denominator by x^3, we get $\lim_{x\to-\infty} \dfrac{x^2/x^3 + 3x/x^3}{7x^3/x^3+7x^2/x^3+3/x^3} = \lim_{x\to-\infty} \dfrac{1/x + 3/x^2}{7+7/x+3/x^3}$

as x approaches $-\infty$, $1/x$, $3/x^2$, $7/x$, $3/x^3$ all approaches 0. so $\lim_{x\to-\infty} \dfrac{x^2+3x}{7x^3+7x^2+3} = \dfrac{0+0}{7+0+0} = 0$

5. $\lim_{x\to-\infty} (x^5-3x^4) = -\infty$

sol: with polynomials, the limit is determined by the highest power x^5 in this case

To be mathematical $\lim_{x\to-\infty} (x^5-3x^4) = \lim_{x\to-\infty} x^5(1-3x^4/x^5)$

$= \lim_{x\to-\infty} x^5(1-3/x) = \lim_{x\to-\infty} x^5(1-0) = \lim_{x\to-\infty} x^5$

since 5 is odd, as x goes to ~~negative~~ $-\infty$, x^5 goes to $-\infty$, so the limit is $-\infty$

we use the same technique in #3 & 4 for rational function

6 find horizontal & vertical asymptotes of

$$y = \frac{9x^2+10}{x^2-10x+21}$$

sol: horizontal asymptote is horizontal, therefore is what happens when $x \to \pm\infty$

$$\lim_{x\to\infty} \frac{9x^2+10}{x^2-10x+21} = \lim_{x\to\infty} \frac{9x^2/x^2+10/x^2}{x^2/x^2-10x/x^2+21/x^2} = \lim_{x\to\infty} \frac{9+10/x^2}{1-10/x+21/x^2} = \frac{9}{1} = 9$$

so the horizontal asymptote is $\underline{\underline{y=9}}$

vertical asymptote is where $y \to \infty$ in another word, since this is a fraction, where the denominator is 0

solving $x^2-10x+21=0 \Rightarrow (x-3)(x-7)=0 \Rightarrow x=3$ and $x=3$

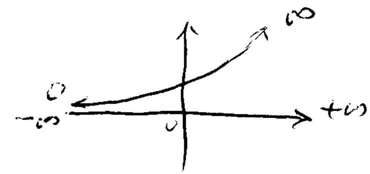
we have the vertical asymptotes are $\underline{\underline{x=3}}$, $\underline{\underline{x=7}}$

7. $\lim\limits_{x\to\infty} e^{-\frac{x}{6}} = 0$

sol: $\lim\limits_{x\to\infty} e^{-\frac{x}{6}} = \lim\limits_{x\to\infty} \frac{1}{e^{x/6}}$ as $x\to\infty$ $x/6 \to \infty$ $e^{x/6}\to\infty$ $\frac{1}{e^{x/6}} \to \underline{0}$

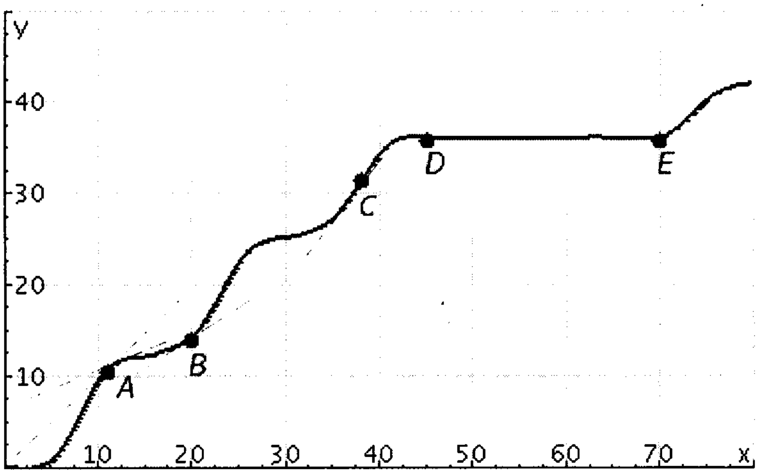
another way to see this is as $x\to\infty$ $-x/6 \to -\infty$ $e^{-\infty} \to 0$

remember the graph of e^x



8. $\lim\limits_{x\to\infty} \frac{12x^2+8x+3}{4x^2+3} = 3$

sol. $\lim\limits_{x\to\infty} \frac{12x^2+8x+3}{4x^2+3} = \lim\limits_{x\to\infty} \frac{12x^2/x^2+8x/x^2+3/x^2}{4x^2/x^2+3/x^2} = \lim\limits_{x\to\infty} \frac{12+8/x+3/x^2}{4+3/x^2} = \frac{12}{4} =$



(a) what was the initial velocity of the car? 0

(b) Was the car going faster at B or at C? C

(c) Was the car slowing down, speeding up or neither at A, B,

: (a) this is the graph of distance. The velocity at ~~the~~ a pt. being the derivative of the distance function, is given graph by the slope of the tangent line

Initially, the tangent line appears to be horizontal so the initial velocity is 0

(b) It appears that the slope of the tangent line at B is smaller than that at C So the car is going faster at C

(c) It appears that at points right after pt. A, the slope of the tangent line is getting smaller. Hence at A, the car is slowing down, similarly at B the car is speeding up. at C it's neither

10. find the slope of the tangent line to $y = x^3$ at $(-2, -8)$

Sol: $m = \dfrac{f(a+h) - f(a)}{h}$ in this case $f(x) = y = x^3$, $a = -2$

so $m = \lim\limits_{h \to 0} \dfrac{(-2+h)^3 - (-2)^3}{h}$

$(-2+h)^3 = (-2+h)(-2+h)(-2+h) = (4 - 4h + h^2)(-2+h)$

$= -8 + 12h - 6h^2 + h^3$

so $m = \lim\limits_{h \to 0} \dfrac{(-8 + 12h - 6h^2 + h^3) - (-8)}{h} = \lim\limits_{h \to 0} \dfrac{12h - 6h^2 + h^3}{h} = \lim\limits_{h \to 0} (12 - 6h + h^2) = 12$

to check $y' = 3x^2$ when $x = -2$ $y'(-2) = 3(-2)^2 = 12$

11. ~~ma~~ find $f'(a)$ when $f(x) = 7 - 7x + 3x^2$

Sol. $f'(a) = m = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

$f(a+h) = 7 - 7(a+h) + 3(a+h)^2 = 7 - 7a - 7h + 3a^2 + 6ah + h^2$

$f(a) = 7 - 7a + 3a^2 \qquad = 7 - 7a \qquad + 3a^2$

$f(a+h) - f(a) = -7h + 6ah + h^2$

$\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h \to 0} \dfrac{-7h + 6ah + h^2}{h} = \lim\limits_{h \to 0} -7 + 6a + h = \underline{\underline{-7 + 6a}}$

to check $f'(x) = -7 + 6x$ so $f'(a) = -7 + 6a$

12 find $f'(a)$ for $f(x) = \frac{9}{\sqrt{9+x}}$

$$f'(a) = m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$f(a+h) - f(a) = \frac{9}{\sqrt{9+a+h}} - \frac{9}{\sqrt{9+a}} = \frac{9\sqrt{9+a} - 9\sqrt{9+a+h}}{\sqrt{9+a+h}\sqrt{9+a}} = \frac{9(\sqrt{9+a} - \sqrt{9+a}}{\sqrt{(9+a+h)(9+a}}$$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{9}{h} \frac{\sqrt{9+a} - \sqrt{9+a+h}}{\sqrt{(9+a+h)(9+a)}}$$

$$= \lim_{h\to 0} \frac{9}{h} \frac{(\sqrt{9+a} - \sqrt{9+a+h})}{\sqrt{(9+a+h)(9+a)}} \cdot \frac{(\sqrt{9+a} + \sqrt{9+a+h})}{(\sqrt{9+a} + \sqrt{9+a+h})}$$

$$= \lim_{h\to 0} \frac{9}{h} \frac{(\sqrt{9+a})^2 - (\sqrt{9+a+h})^2}{\sqrt{(9+a+h)(9+a)}\,(\sqrt{9+a} + \sqrt{9+a+h})} = \lim_{h\to 0} \frac{9}{h} \frac{(9+a) - (9+a+h)}{\text{same denominator}}$$

$$= \lim_{h\to 0} \frac{9}{h} \frac{-h}{\text{same den}} = \lim_{h\to 0} \frac{-9}{\sqrt{(9+a+h)(9+a)}\,(\sqrt{9+a} + \sqrt{9+a+h})} = \frac{-9}{\sqrt{(9+a)^2}\,(2\sqrt{9+a})}$$

$$= -\frac{9}{2} \frac{1}{\sqrt{(9+a)^3}} = -\frac{9}{2}(9+a)^{3/2}$$

13. $\lim_{h\to 0} \frac{(1+h)^9 - 1}{h}$ is the derivative for what f & a

sol. $m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ in this case $= \frac{(1+h)^9 - 1}{h}$

comparing the numerator one sees that if we choose $f(x) = x^9$ (~~since~~ because of the first term) then set $a=1$ the two expressions agree. so $f(x) = x^9$, $a=1$ is a solution.

on the other hand we can also set $f(x) = (1+x)^9$ this also agrees with the first term in the numerator, then a has to be 0. so $f(x) = (1+x)^9$, $a=0$ is another solution.

14. find the equation of the tangent line of $y = \frac{2x}{(x+1)^2}$ at $(0,0)$

sol: $y' = \left(\frac{2x}{(x+1)^2}\right)' \overset{\text{quotient rule}}{=} \frac{(2x)'(x+1)^2 - 2x \cdot [(x+1)^2]'}{(x+1)^4} = \frac{2(x+1)^2 - 2x \cdot 2(x+1)}{(x+1)^4}$

take out common factor $2(x+1)$ $\frac{2(x+1)[x+1-2x]}{(x+1)^4} = \frac{2(x+1-2x)}{(x+1)^3} = \frac{2(1-x)}{(x+1)^3}$

plugging in $x=0$ $y'(0) = \frac{2}{1^3} = 2$

So the slope of the tangent line $m = 2$

$y = 2x + b$ is the equation of the tangent line at $(0,0)$ to find b, plugging in $(0,0)$ for x, y

$b = 0$ so the equation of the tangent line is $\underline{\underline{y=2x}}$

15. find the equation of the tangent line to $y = x^3 - 7x + 8$ at the point $(1,2)$, $(2,2)$

sol. the slope of the tangent line $m = y' = 3x^2 - 7$

plugging in $x=1$ we get $m = y'(1) = 3 \cdot 1^2 - 7 = -4$

equation of the tangent line is $y = -4x + b$

plugging in $x=1$, $y=2$, solving for $b = 2+4 = 6$

So at $(1,2)$ the equation of the tangent line is

$\underline{\underline{y = -4x + 6}}$

at $(2,2)$ plugging in $x=2$ to $m = y' = y'(2) = 3 \cdot 2^2 - 7 = 5$

plugging in $x=2$, $y=2$ to $y = 5x + b$ solve $b = 2 - 5 \cdot 2 = -8$

at $(2,2)$ eq. of the tangent line is $\underline{y = 5x - 8}$

16. If a ball is thrown in the air with an initial velocity of 80ft/s then its height (in feet) after t seconds is given by $y = 80t - 16t^2$. Find the velocity when $t = 3$

sol. we know the height function (position function) is $y = 80t - 16t^2$. the velocity function is

$$v(t) = y' = 80 - 32t \quad \text{so when } t = 3$$

$$v(3) = 80 - 32 \cdot 3 = 80 - 96 = -16 \text{ ft/s}$$

17 If an arrow is shot up on the moon with an initial velocity of 40m/s its height after t seconds is $H = 40t - 0.83t^2$ when will the arrow hit the moon?

sol. the arrow will hit the moon when $H = 0$

so $40t - 0.83t^2 = 0$ solve $(40 - 0.83t)t = 0$ so $t = 0$ or $\frac{40}{0.83} = 48$

$t = 0$ is when the arrow is just beginning to take off

$t = 48.19$ is when the arrow falls back and hit the ground

18. Same as #17 with initial velocity 59m/s. $H = 59t - 0.83t^2$ with what velocity will the arrow hit the moon?

sol. velocity $= H' = 59 - 2 \cdot 0.83t = 59 - 1.66t$

what's t when the arrow hit the moon (refer to #17)

$H = 0 \Rightarrow 59t - 0.83t^2 = 0 \Rightarrow t = \frac{59}{0.83} = 71.08$

so $v(71.08) = 59 - 1.66 \cdot 71.08 = -59$ which makes sense because by symmetry.

19. if $f(t) = t^4 - 5t$ find $f'(a)$

$f'(t) = 4t^3 - 5$ so $f'(a) = 4a^3 - 5$ (plugging in a for t)

20. the life expectance is given by the following table

t	$E(t)$	t	$E(t)$
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	74.1
1950	65.6		

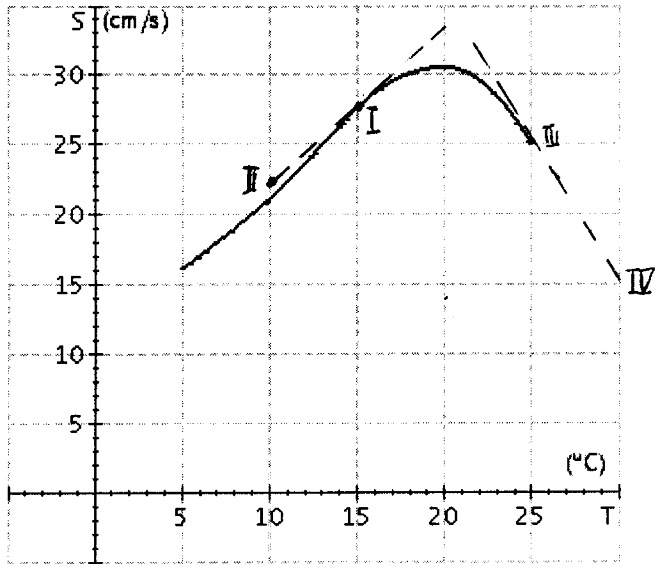
Find the value of $E'(1930)$ and $E'(1950)$

sol: $E'(1930) = m \approx \dfrac{f(a+h) - f(a)}{h}$ here $a = 1930$ $f = E(t)$, $h = 10$ yrs

$$E'(1930) \approx \frac{E(1940) - E(1930)}{10} = \frac{62.5 - 57.4}{10} = 0.51$$

$$E'(1950) \approx \frac{E(1960) - E(1950)}{10} = \frac{66.6 - 65.6}{10} = 0.1$$

21



$S'(15) = 1$

$S'(25) = -2$

Sol: I drew the tangent lines at $T=15$ & $T=25$ as shown in the graph. To compute the slopes

I estimate the coordinates for points I $(15, 27.5)$

II $(10, 22.5)$, III $(25, 25)$ IV $(30, 15)$

Using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$S'(15) = \frac{27.5 - 22.5}{15 - 10} \doteq 1$$

$$S'(25) = \frac{15 - 25}{30 - 25} \doteq -2$$