Math 123 Solutions to Exam 1, Zeppo version

2 points

1. What is the value of $\cos\left(\frac{3\pi}{4}\right)$?

Solution: The reference angle is $\pi/4$, which has a cosine of $1/\sqrt{2}$. $3\pi/4$ is in the second quadrant, so the cosine is negative.

1. $-\frac{1}{\sqrt{2}}$

2 points

2. What is the largest domain on which the function $f(x) = \dfrac{3-x}{\sqrt{5-x}}$ is defined?

Solution: Note that $\sqrt{5-x}$ is only defined for $x \leq 5$. But if $x = 5$, the denominator is zero. So we need $x < 5$.

2. $x < 5$

2 points

3. If $f(y) = y^2 - 3y$ and $g(x) = 2\sqrt{x}$, find $f(g(25))$.

Solution: $f(g(25)) = f(2\sqrt{25}) = f(10) = 100 - 30 = 70$.

3. 70

2 points

4. Suppose $f(x) = \begin{cases} x^2 - 16 & \text{if } x \leq 1 \\ 2x - 5 & \text{if } x > 1 \end{cases}$. Find all x such that $f(x) = 0$.

Solution: If $x \leq 1$, we look at $x^2 - 16 = 0$, so $x = \pm 4$. Since $x \leq 1$, we only use -4. For $x > 1$, we solve $2x - 5 = 0$ to get $x = 5/2$ (which is greater than 1).

4. $x = -4, \quad x = 5/2$

2 points

5. Suppose that $\tan\alpha = \dfrac{10}{9}$ and $\pi < \alpha < \dfrac{3\pi}{2}$. Find $\sin\alpha$.

Solution: Since $\tan\alpha = \frac{10}{9}$, by the Pythagorean Theorem the hypotenuse of a relevant triangle is $\sqrt{181}$, so the sine of the reference angle is $\frac{10}{\sqrt{181}}$. $\sin\alpha$ is negative in the 3rd quadrant.

5. $-\dfrac{10}{\sqrt{181}}$

2 points

6. Let $f(A) = A^2 + 5$. Find $f(x+h) - f(x)$; simplify your answer as much as possible.

Solution: $f(x+h) = (x+h)^2 + 5 = x^2 + 2xh + h^2 + 5$, and $f(x) = x^2 + 5$.
Thus, $f(x+h) - f(x) = (x^2 + 2xh + h^2 + 5) - (x^2 + 5) = 2xh + h^2$.

6. $2xh + h^2$

2 points

7. Give two angles x with $0 \leq x \leq \pi$ for which $\tan(2x) = \dfrac{\sqrt{3}}{3}$.

Solution: Since $\tan(2x) = \frac{\sqrt{3}}{3}$, first consider $\tan\theta = \frac{\sqrt{3}}{3}$, which gives $\theta = \pi/3$ or $\theta = 7\pi/6$. Since $\theta = 2x$, divide each by 2.

7. $\pi/12, \quad 7\pi/12$

2 points

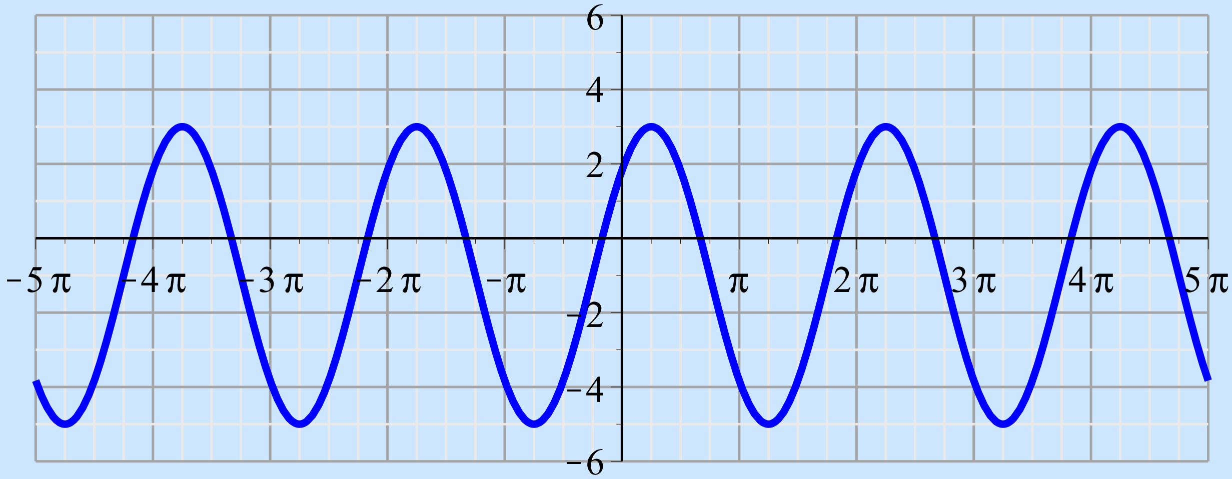
8. Suppose $f(x) = 4x^3 + 5$. Find $f^{-1}(x)$, if it exists. (If it does not exist, write DNE).

Solution: Take $y = 4x^3 + 5$, solve for x. $y - 5 = 4x^3$,
so $(y-5)/4 = x^3$. Take the cube root of both sides: $x = \sqrt[3]{\frac{y-5}{4}}$.
Now substitute y for x.

8. $f^{-1}(x) = \sqrt[3]{\dfrac{x-5}{4}}$

$\boxed{\text{8 points}}$ 9. On the axes provided below, sketch the graph of $4\sin\left(x+\frac{\pi}{4}\right)-1$.

Solution: This is the graph of the sine, but with the graph shifted $\pi/4$ to the left, and scaled vertically by a factor of 4 and moved 1 unit down.



$\boxed{\text{8 points}}$ 10. Find the smallest positive value of x so that $4\sin\left(x+\frac{\pi}{4}\right)-1=1$.

Solution: Adding 1 to both sides of the above gives us

$$4\sin\left(x+\frac{\pi}{4}\right)=2.$$

After dividing both sides by 4, we have

$$\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{2}.$$

We know that $\sin\theta=1/2$ gives $\theta=\pi/6$, so we would need to have

$$x+\frac{\pi}{4}=\frac{\pi}{6}, \qquad \text{or} \qquad x=\frac{\pi}{6}-\frac{\pi}{4}=-\frac{\pi}{12}.$$

Unfortunately, this is a negative number, so we need to think again.

Of course, $\sin(5\pi/6)=1/2$ as well, so we have

$$x+\frac{\pi}{4}=\frac{5\pi}{6}, \qquad \text{or} \qquad x=\frac{5\pi}{6}-\frac{\pi}{4}=\frac{7\pi}{12}.$$

This is indeed the smallest positive solution. Note that this solution agrees with the graph above, since the line $y=1$ crosses the graph at $x=\frac{7\pi}{12}$, and this is the first crossing to the right of the y-axis.

11. A swimming pool holds 50,000 gallons of water. Initially the pool is empty and it will filled using three different types of hoses. First, a hose that pumps pure water at a rate of 500 gallons per hour begins to fill the pool. When the volume of water in the pool is 10,000 gallons, the second hose is used in addition to the first hose. The second hose pumps slightly chlorinated water at 300 gallons per hour. When the volume of water in the pool is 30,000 gallons, the third hose is used in addition to the first two hoses. The third hose pumps water treated with an antibiotic at a rate of 200 gallons per hour.

8 points (a) Write an expression for the function $V(t)$ which represents the volume of water at time t (in hours after the water was turned on).

Solution: Since the pool is initially empty, we have $V(0) = 0$. (If you like, you can write that $V(t) = 0$ for $t < 0$ or not; it is a matter of taste.) Initially, the water comes in at 500 gallons per hour, so $V(t) = 500t$ for t small.

We have to figure out how long it takes to get 10000 gallons in the pool; that will be 10000/500, or 200 hours. This means $V(t) = 500t$ from when the pool starts filling ($t = 0$) until it has 10000 gallons ($t = 20$).

Then the next hose kicks in, adding an additional 300 gal/hr until there are 30000 gal in the pool. That means that we are filling at a total rate of 800 gal/hr, and it takes $20000/800 = 25$ hrs to get the additional 20000 gallons needed until the third hose begins. We need to measure the time since the first hose came on, which is $t - 20$. The contribution to the initial 10000 gallons during this time period is $800(t - 20)$.

After 45 hours of filling, we will have 30000 gallons, and begin filling at a rate of 1000 gallons per hour (because of the third hose). To get the remaining 20000 gallons requires and additional 20 hours, or 65 hours until the pool is full.

Putting it all together, we have

$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 500t & \text{if } 0 \le t < 20 \\ 10000 + 800(t-20) & \text{if } 20 \le t < 45 \\ 30000 + 1000(t-45) & \text{if } 45 \le t < 65 \\ 50000 & \text{if } 65 \le t \end{cases}$$

4 points (b) How long will it take (in hours) until the pool will be half-full (that is, until it contains 25,000 gallons of water)?

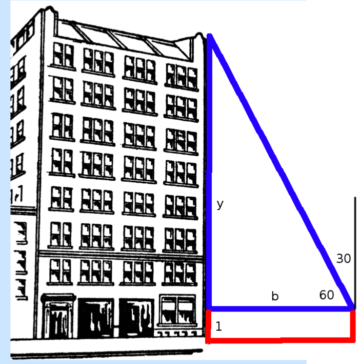
Solution: Notice that $10000 < 25000 < 30000$, so $20 < t < 45$. This means we need to solve

$$10000 + 800(t - 20) = 25000$$

This gives us $800(t - 20) = 15000$, or (after dividing both sides by 800), $t - 20 = 75/4$, so $t = 20 + 75/4 = 155/4 = 38.75$ hours. The pool is half-full after 38.75 hours.

8 points 12. You are standing 30 meters from the base of a tall building, and you aim a laser pointer at the closest part of the top of the building. You measure that the laser pointer is $30°$ tilted from pointing straight up. You are holding the laser pointer 1 meter off the ground. How tall is the building?

Solution:
Think of the line from the laser pointer to the top of the building as the hypotenuse of a right triangle. Since the laser pointer is $30°$ from straight up, the angle it makes with the base is $60°$. The base is 30 meters, and the building forms the other leg. Let the height of the building above the laser pointer be y, so the building will be $y+1$ meters tall. Since it is a right triangle, $\tan(60°) = \frac{y}{30}$, that is, the height of the building is



$$y + 1 = 1 + 30\tan 60° = 1 + 30\sqrt{3} \approx 52.9 \text{ meters tall}$$

8 points 13. Let $f(x) = \dfrac{4x-7}{2x+1}$. Find $f^{-1}(x)$.

Solution: We write $y = \dfrac{4x-7}{2x+1}$ and solve for x. Thus, we have

$$\begin{aligned} y &= \frac{4x-7}{2x+1} \\ (2x+1)y &= 4x-7 \\ 2xy + y &= 4x - 7 \\ 2xy - 4x &= -y - 7 \\ x(2y-4) &= -(y+7) \\ x &= -\frac{y+7}{2y-4} = \frac{y+7}{4-2y} \end{aligned}$$

Thus,

$$f^{-1}(x) = \frac{x+7}{4-2x}$$