MAT 118 Spring 2017

Practice Final Exam

1. [17 pts] Consider the following preference schedule for an election between candidates A, B, C:

# voters:	6	1	2	3
1st	A	C	B	C
2nd	B	B	C	A
3rd	C	A	A	B

(a) Which candidate wins using the plurality method?

A wins with 6 votes.

(b) Use the Borda count method to determine the outcome.

We assign 3 pts to each 1st place vote, 2 pts to 2nd place votes, and 1 pt to each 3rd place vote.

Pts for A: $3 \times 6 + 2 \times 3 + 1 \times 3 = 18 + 6 + 3 = 27$.

Pts for B: $3 \times 2 + 2 \times 7 + 1 \times 3 = 6 + 14 + 3 = 23$.

Pts for C: $3 \times 4 + 2 \times 2 + 1 \times 6 = 12 + 4 + 6 = 22$.

Thus A wins.

(c) Use the plurality-with-elimination method to determine the outcome.

First round: A has 6 1st place votes, B has 2, and C has 4. No one has a majority, so we eliminate B, the candidate with the fewest votes. In the column with B's 1st place votes, C is next, so these 2 votes go to C. Next round: A has 6, C has 6. We stop here with A and C tied.

(d) Finally, apply the method of pairwise comparisons to determine the outcome.

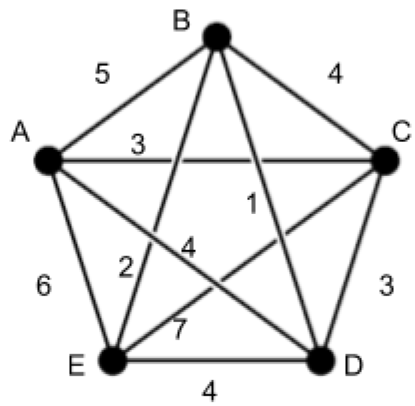
A v B: 9 voters prefer A over B, while the remaining 3 prefer B over A. So A gets 1 pt.

A v C: this comparison is a tie. Each of A and C are awarded ½ a pt.

B v C: 8 voters prefer B over C, while only 4 prefer C over B. So B gets 1 pt.

In total, A has 1+½ pts, B has 1 pt, and C has ½ a pt. Thus A wins.

2. Use NNA starting at C to find a Hamilton circuit for the following, and give its total weight.



There are 2 answers: C, A, D, B, E, C with total weight 3 + 4 + 1 + 2 + 7 = 17;
and C, D, B, E, A, C with total weight 3 + 1 + 2 + 6 + 3 = 15.

3. Consider the weighted voting system [19; 9, 8, 1, 1].

(a) List the sequential coalitions and underline the pivotal players. Then compute the Shapley-Shubik power distribution.

Here are the sequential coalitions:

$\langle P_1, P_2, P_3, \underline{P_4} \rangle$	$\langle P_4, P_2, P_3, \underline{P_1} \rangle$	$\langle P_4, P_1, P_3, \underline{P_2} \rangle$	$\langle P_4, P_1, P_2, \underline{P_3} \rangle$
$\langle P_1, P_3, P_2, \underline{P_4} \rangle$	$\langle P_4, P_3, P_2, \underline{P_1} \rangle$	$\langle P_4, P_3, P_1, \underline{P_2} \rangle$	$\langle P_4, P_2, P_1, \underline{P_3} \rangle$
$\langle P_2, P_1, P_3, \underline{P_4} \rangle$	$\langle P_2, P_4, P_3, \underline{P_1} \rangle$	$\langle P_1, P_4, P_3, \underline{P_2} \rangle$	$\langle P_1, P_4, P_2, \underline{P_3} \rangle$
$\langle P_2, P_3, P_1, \underline{P_4} \rangle$	$\langle P_2, P_3, P_4, \underline{P_1} \rangle$	$\langle P_1, P_3, P_4, \underline{P_2} \rangle$	$\langle P_1, P_2, P_4, \underline{P_3} \rangle$
$\langle P_3, P_1, P_2, \underline{P_4} \rangle$	$\langle P_3, P_4, P_2, \underline{P_1} \rangle$	$\langle P_3, P_4, P_1, \underline{P_2} \rangle$	$\langle P_2, P_4, P_1, \underline{P_3} \rangle$
$\langle P_3, P_2, P_1, \underline{P_4} \rangle$	$\langle P_3, P_2, P_4, \underline{P_1} \rangle$	$\langle P_3, P_1, P_4, \underline{P_2} \rangle$	$\langle P_2, P_1, P_4, \underline{P_3} \rangle$

Pivotal counts are $SS_1 = SS_2 = SS_3 = SS_4 = 6$. Since there are 4! = 24 sequential coalitions, we get $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 6/24 = ¼$.

(b) List the winning coalitions and underline the critical players. Using this information compute the Banzhaf power distribution.

The only winning coalition is $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, \underline{P_4}\}$ and all players are critical. Thus the critical counts are $B_1 = B_2 = B_3 = B_4 = 1$, the total is T = 4, and we get $\beta_1 = \beta_2 = \beta_3 = \beta_4 = ¼$.

Remark: As mentioned in class (and apparent from the solution above), the weighted voting system [19; 9, 8, 1, 1] is rather uninteresting. Consider the alternative problem using [10; 9, 8, 1]. Then

$$\langle P_1, \underline{P_2}, P_3 \rangle$$
$$\langle P_1, \underline{P_3}, P_2 \rangle$$
$$\langle P_2, \underline{P_1}, P_3 \rangle$$
$$\langle P_2, P_3, \underline{P_1} \rangle$$
$$\langle P_3, \underline{P_1}, P_2 \rangle$$
$$\langle P_3, P_2, \underline{P_1} \rangle$$

so that $SS_1 = 4$, $SS_2 = SS_3 = 1$. There are 3! = 6 sequential coalitions, so $\sigma_1 = 4/6 = 2/3$, $\sigma_2 = \sigma_3 = 1/6$.

For Banzhaf power: we have the winning coalitions $\{ \underline{P_1}, P_2, P_3 \}$, $\{ \underline{P_1}, \underline{P_2} \}$, $\{ \underline{P_1}, \underline{P_3} \}$ with critical players underlined. We find $B_1 = 3$, $B_2 = 1$, $B_3 = 1$, and $T = 3 + 1 + 1 = 5$. Thus $\beta_1 = 3/5$, $\beta_2 = \beta_3 = 1/5$.

4. Consider the sequence 4, 9, 14, 19, …

(a) Is this sequence arithmetic, or geometric?
The differences 9 - 4 = 5, 14 - 9 = 5, 19 - 14 = 5 are constant (d = 5), so it is arithmetic.

(b) Compute the sum of the first 100 terms of the sequence.
We use the arithmetic sum formula, which is given by:

$$P_0 + P_1 + \ldots + P_{N-1} = (P_0 + P_{N-1})N/2$$

The first 100 terms are terms $P_0, P_1, \ldots, P_{99}$ so we will take N-1=99 and N=100.
First, $P_{99} = P_0 + 99d = 4 + 99(5) = 4 + 495 = 499$. Then

$$P_0 + P_1 + \ldots + P_{99} = (P_0 + P_{99})(100)/2 = (4 + 499)(50) = 25150.$$

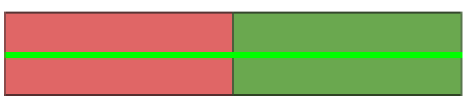
5. Abe and Barb are dividing a $9 sandwich which is ½ chicken parm and ½ vegetarian. Barb likes the chicken parm part 2 times as much as the vegetarian part, while Abe is vegetarian.

(a) Make a table that records how much money the different flavor halves of the sandwich are worth to Abe and Barb, in dollars.

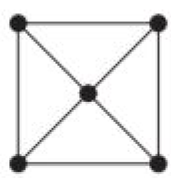
	CP	V
A	$0	$9
B	$6	$3

(b) Draw and describe a cut that Abe can make if he is acting as the divider in the divider-chooser method.

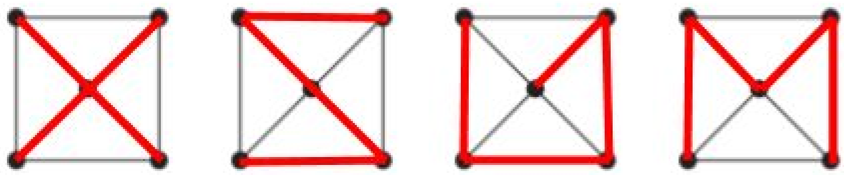
Either of the following is fine:



6. Draw 4 spanning trees of the following graph:



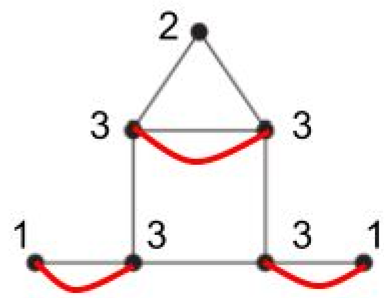
Here are 4 (there are many more):



7. An exponentially growing population sequence has initial $P_0 = 2$ and $R = 3$. What is P_5?

In general, $P_N = R^N P_0$. Thus $P_5 = 3^5(2) = 486$.

8. On the following graph, draw the degrees of each vertex, and then give an eulerization.



9. In a logistic growth model, $r = ½$. If the carrying capacity is 100, is there a non-zero population amount that can be in equilibrium (i.e. $p_0 = p_1 = p_2 = ...$)?

As we saw in class, for equilibrium to occur, either the population is zero or $p_0 = 1\text{-}1/r$. Here $r = ½$, so we would need $p_0 = 1 - 1/(½) = 1\text{-}2 = \text{-}1$. However, a p-value is a percentage, and so must be a number between 0 and 1. So $p_0 = \text{-}1$ is nonsense, and the answer is “NO”!

Remark: The derivation of $p_0 = 1 - 1/r$ is very straightforward. The general logistic equation says

$$p_{N+1} = r(1 - p_N)p_N$$

and so if we are to have equilibrium ($p_0 = p_1 = ….$) then in particular we have

$$p_0 = p_1 = r(1 - p_0)p_0$$

Now, if p_0 is nonzero, then dividing both sides by p_0 yields $1 = r(1 - p_0)$, and solving for p_0 yields the equation $p_0 = 1 - 1/r$, which is what was used above.