MAT 118 Spring 2017

Midterm #1

Name ______________________________

ID# ______________________________

Please show your work.
And remember: no calculators!
The test is out of 100 points.

1. [17 pts] Consider the following preference schedule for an election between candidates A, B, C:

# voters:	6	3	1	2
1st	A	C	B	C
2nd	B	B	C	A
3rd	C	A	A	B

(a) [2 pts] Which candidate wins using the plurality method?
A wins, since A has the most 1st place votes (6).

(b) [5 pts] Use the weighted Borda count method to determine the outcome, in which 1st place votes are 5 points, 2nd place votes are 4 points, and 3rd place votes are 1 point.
Since A has 6 first place votes, 2 second place votes, and 4 third place votes, A receives:
$6 \times 5 + 2 \times 4 + 4 \times 1 = 30 + 8 + 4 = 42$. Similarly, the number of points that B receives is:
$1 \times 5 + 9 \times 4 + 2 \times 1 = 5 + 36 + 2 = 43$. And finally for C we compute:
$5 \times 5 + 1 \times 4 + 6 \times 1 = 25 + 4 + 6 = 35$.
Thus B is 1st, A is 2nd, and C is 3rd.

(c) [5 pts] Use the plurality-with-elimination method to determine the outcome.

In the first round, we compare the candidates using 1st place votes: A has 6, B has 1, And C has 5. No one has a majority (which would be at least 7 votes out of the total 12), so we eliminate the candidate with the fewest votes, which here is B. We see that the 1 vote that B had has C as the 2nd preference, so this 1 vote gets transferred to C.
So in round two, A still has 6 votes while now C has 5 + 1 = 6. Thus A and C tie.

(d) [5 pts] Finally, apply the method of pairwise comparisons to determine the outcome.

There are three possible pairwise comparisons: A v B, B v C and A v C.
A v B: 6 + 2 = 8 voters prefer A over B, while 3 + 1 = 4 prefer B over A.
Thus A wins in the pairwise comparison A v B, and is awarded 1 point.
B v C: 6 + 1 = 7 voters prefer B over C, while 3 + 2 = 5 prefer C over B.
Thus B wins in a comparison to C, and is awarded 1 point.
A v C: 6 voters prefer A over C, while 3 + 1 + 2 = 6 prefer C over A.
This is a tie, and each of A and C receive 1/2 a point.
In summary, A gets 3/2 points, B gets 1 point, and C gets 1/2 a point.
The candidates are thus ranked from 1st to last in the order A, B , C.

2. [5 pts] Is there a Condorcet candidate in the following election? If so, who is it? Even if your answer is “no,” be sure to show the work that substantiates your claim.

# voters:	6	5	4
1st	A	C	B
2nd	B	D	A
3rd	C	A	C
4th	D	B	D

To answer this we must do some pairwise comparisons.
A v B: 6 + 5 = 11 voters prefer A over B, while 4 prefer B over A. So A wins.
A v C: 6 + 4 = 10 voters prefer A over C, while 5 prefer C over A. So A wins.
A v D: 6 + 4 = 10 voters prefer A over D, while 5 prefer D over A. So A wins.
So yes, there is a Condorcet candidate, and it is A.

3. [13 pts] Consider the weighted voting system [17; 7, 6, 4, 3] with players P_1, P_2, P_3, P_4.

(a) [5 pts] Does any player have veto power? If so, how many and which ones?

Note the total votes in the system: $V = 7 + 6 + 4 + 3 = 20$. The quota $q = 17$.
For a player with weight w to have veto power means that $V - w < q$.
P_1 has veto power because $20 - 7 = 13 < 17$.
P_2 has veto power because $20 - 6 = 14 < 17$.
P_3 has veto power because $20 - 4 = 16 < 17$.
These three are the only ones:
P_4 does *not* have veto power since $20 - 3 = 17$ is not less than 17.

(b) [5 pts] List the winning coalitions and underline the critical players. Using this information compute the Banzhaf power distribution.

The winning coalitions are $\{ \underline{P_1}, \underline{P_2}, \underline{P_3}, P_4 \}$ and $\{ \underline{P_1}, \underline{P_2}, \underline{P_3} \}$.
The critical players are underlined.
The critical counts are $B_1 = B_2 = B_3 = 2$ and $B_4 = 0$.
The total critical count is $T = 2 + 2 + 2 + 0 = 6$.
The BPI's are then $\beta_1 = \beta_2 = \beta_3 = 2/6 = 1/3$ and $\beta_4 = 0/6 = 0$.

(c) [3 pts] How many sequential coalitions are there in this weighted voting system?

There are $N = 4$ players, so there are
$N! = 4! = 4 \times 3 \times 2 \times 1 = 24$ sequential coalitions.

4. [10 pts] Consider the weighted voting system [500; 499, 498, 1].

(a) [5 pts] List the sequential coalitions and underline the pivotal players. Then compute the Shapley-Shubik power distribution.

Here are the sequential coalitions, with pivotal players underlined:

$\langle P_1, \underline{P_2}, P_3 \rangle$
$\langle P_1, \underline{P_3}, P_2 \rangle$
$\langle P_2, \underline{P_1}, P_3 \rangle$
$\langle P_2, P_3, \underline{P_1} \rangle$
$\langle P_3, \underline{P_1}, P_2 \rangle$
$\langle P_3, P_2, \underline{P_1} \rangle$

Thus the pivotal counts are $SS_1 = 4$ and $SS_2 = SS_3 = 1$. The Shapley-Shubik indices are given by these numbers divided by $N! = 3! = 6$, and we get $\sigma_1 = 4/6 = 2/3$ and $\sigma_2 = \sigma_3 = 1/6$.

(b) [5 pts] List the winning coalitions and underline the critical players. Using this information compute the Banzhaf power distribution.

Here are the winning coalitions, with critical players underlined:

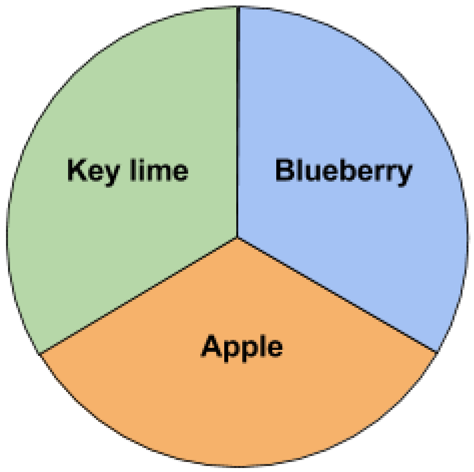
$\{ \underline{P_1}, P_2, P_3 \}$
$\{ \underline{P_1}, \underline{P_2} \}$
$\{ \underline{P_1}, \underline{P_3} \}$

The critical counts are $B_1 = 3$ and $B_2 = B_3 = 1$.
The total critical count is $T = 3 + 1 + 1 = 5$.
The BPI's are then $\beta_1 = 3/5$ and $\beta_2 = \beta_3 = 1/5$.

5. [6 pts] In a weighted voting system with 3 players, the winning coalitions are $\{ P_1, P_2 \}$, $\{ P_1, P_3 \}$ and $\{ P_1, P_2, P_3 \}$. Compute the Shapley-Shubik power distribution.

This is the same data as in 4(b). So the answer is 4(a).

6. [13 pts] Albert and Brenda are dividing a massive multi-flavored pie using the divider-chooser method. After flipping a coin, it is decided that Albert will be the divider. The pie has three flavors: apple, blueberry and key lime. Here's an illustration of the pie:

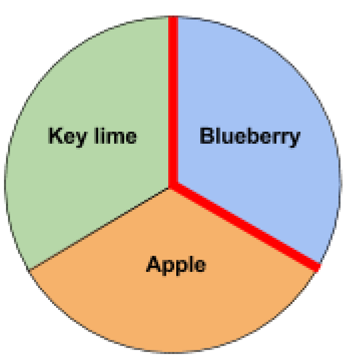


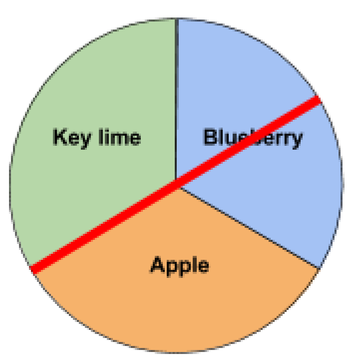
Albert likes apple and key lime equally, but likes blueberry twice as much as either of those other flavors. Brenda likes all three flavors equally.

(a) [3 pts] Make a table listing the values, relative to the total value of the cake, of each flavor according to each of Albert and Brenda. Use either fractions or percentages.

	KL	B	A
A	1/4	1/2	1/4
B	1/3	1/3	1/3

(b) [5 pts] Describe, with pictures and in words (i.e. fractions of flavors), **two** distinct ways that Albert can divide the pie. Remember that the division must be into two parts that Albert considers of equal value.





(c) [5 pts]Describe, for each of the two divisions you chose in part (b), the pieces that Brenda will choose, and how much they are worth to her.
For the division on the left, Brenda takes the part containing all the key lime and all the apple. For the division on the right, she takes either half.

7. [13 pts] In problem #6, suppose that we have an additional third player, Chandler. Chandler only likes blueberry. Now suppose the 3 players decide to use the Lone-divider method.

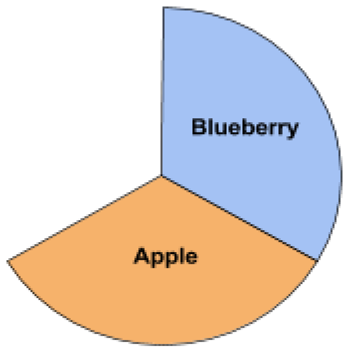
(a) [3 pts] Write a table, enlarging the table from 6(a), listing the values, relative to the total value of the pie, of each flavor according to the 3 players.

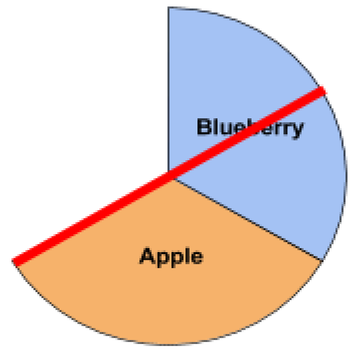
	KL	B	A
A	1/4	1/2	1/4
B	1/3	1/3	1/3
C	0	1	0

(b) [5 pts] Suppose Brenda is the divider, and that she divides the cake along the flavor lines, so that one piece is all apple, one is all blueberry, and one is all key lime. What are the bidding lists (i.e. fair share lists) of Albert and Chandler?
Each of Albert and Chandler only have the blueberry piece on their bidding lists.

(c) [5 pts] Describe how this instance of the Lone-divider method might be completed.

Albert and Chandler want the same piece, so we give Brenda either of the other two pieces, say key lime. Then we recombine blueberry and apple, as in the left picture:

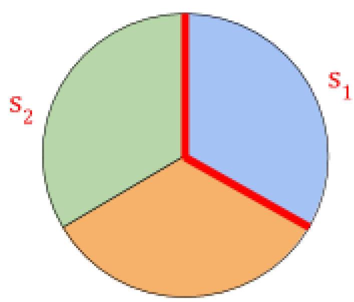


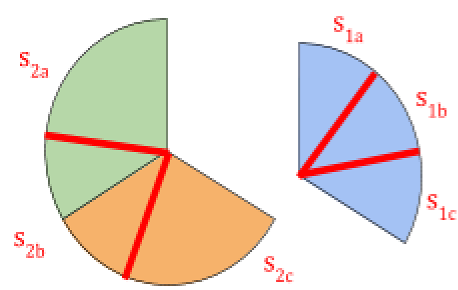


Now Albert and Chandler use the divider-chooser method on this. Say Chandler divides. He can make the cut depicted in the above right picture. Indeed, each part is worth 1/2 to him, since he only like blueberry. Then Albert takes the larger part on bottom, since he still values apple a bit, and Chandler is left with the top blueberry part.

8. [10 pts] Describe how the Lone-Chooser method may go with the pie in #6, in which Chandler is the chooser, and Albert is the divider when Albert and Brenda use the divider-chooser method at the start. How much value does each player end up with?

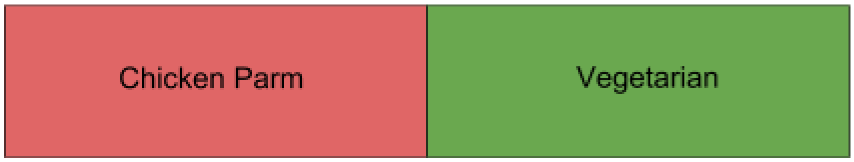
As described in #6, Albert and Brenda can use the divider-chooser method to make the division depicted below on the left, and Albert leaves with the blueberry part, called s_1 and Brenda the key lime and apple part, called s_2.





Then Albert and Brenda divide each of their shares into three parts that they deem equal. This is depicted in the above right picture. Chandler takes (for example) s_{1a} and s_{2a}. The latter piece is not blueberry and is worth nothing to him, while s_{1a} is 1/3 of the blueberry. So Chandler walks away with ⅓ value. Albert is left with s_{1b} and s_{1c}, which is 2/3 of the value of blueberry to him (= 1/2). Multiplying these two fractions tells us that Albert walks away with 1/3. Finally, Brenda is left with s_{2b} and s_{2c}. These two make up 2/3 of 2/3 of the pie; multiplying these two fractions says that Brenda leaves with 4/9 value.

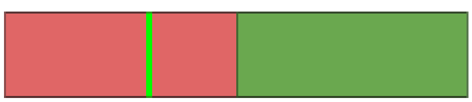
9. [13 pts] Alice and Bert are dividing a $10 sandwich which is half chicken parm and half vegetarian. Bert likes the chicken parm part four times as much as vegetarian part, while Alice is a strict vegetarian.

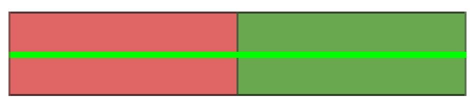


(a) [3 pts] Make a table that records how much money the different flavor halves of the sandwich are worth to Alice and Bert, in dollars.

	CP	V
A	$0	$10
B	$8	$2

(b) [5 pts] Draw and describe a cut that Bert can make if he is acting as the divider in the divider-chooser method.
Bert can either cut leaving 5/8 of the Chicken Parm on the left and 3/8 of the Chicken Parm and all the Vegetarian on the right (see the left hand picture below), or he can cut horizontally (see the right hand picture below).





(c) [5 pts] After the cut, how much money is the piece that Alice chooses worth to her?
In the first cut above, Alice would take the right side piece, which has all the vegetarian included in it, so it is worth $10 to her. In the second cut above, she would take either top or bottom, which each have half the vegetarian, and are each worth $5 to her.