MAE 501 notes

Deanna Dupuis

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Polynomials as functions vs algebraic objects

What is the difference between $\mathbb{Q}[x]$ (polynomials in x and $\mathbb{Q}[\pi]$ (polynomials in π)? The difference is that the elements of $\mathbb{Q}[x]$ are funtions we can evaluate and that the elements of $\mathbb{Q}[\pi]$ are numbers. For each element $f(x)$ of $\mathbb{Q}[x]$, we can get an element of $\mathbb{Q}[\pi]$ by evaluating f at π, that is, calculating $f(\pi)$.

However, while we can evaluate these polynomials, we can also treat them like numbers (without evaluating) in that we can add, multiply, and factor them. Middle School and High School students tend to focus only on the evalutaion aspect of polynomials (that is, polynomials as funtions), and they tend to be unaware of how the polynomials also act like numbers.

For example, if a student were to see $x^2 - 4 = (x+2)(x-2)$. They would most likely only consider the graph of that funtion rather than that it factors much like we can factor $6 = 2 \times 3$.

Students are mostly 1-dimensional when it comes to thinking about polynomials. In the upper levels we tend to forget about evaluation of polynomials and instead think about the ring of polynomials and ideas like irreducibility, ideals, etc. Our goal is to be able to translate between the two worlds easily.

Roots of Polynomials

One of the things we are concerned with in high school is finding roots. Given a polynomial, say $x^4 - \frac{2}{3}x^2 + x + 2$ we want to find its rational roots. We know the only possible candidates are divisors of 2. Why?

Another example: $5x^4 - 2x^2 + 3x + 6$. The only possible roots are:

$$\{\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}\}.$$

Again, why is this? Let's prove a theorem.

Theorem (The Rational Root Theorem). *If $p(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ and if $p\left(\frac{r}{s}\right) = 0$ where $r, s \in \mathbb{Z}, gcd(r,s) = 1$ then r is a divisor of a_0 and s is a divisor of a_d.*

Proof. If $p\left(\frac{r}{s}\right) = 0$ then

$$a_d\left(\frac{r}{s}\right)^d + a_{d-1}\left(\frac{r}{s}\right)^{d-1} + \cdots + a_0 = 0.$$

Multiply through by s^d and we have

$$a_d r^d + a_{d-1} r^{d-1} s + \cdots + a_0 s^d = 0$$

On one hand, subtracting everything except $a_d r^d$, we have

$$a_d r^d = -a_{d-1} r^{d-1} s - \cdots - a_0 s^d$$

and factoring out an s gives

$$a_d r^d = s\left(-a_{d-1} r^{d-1} - \cdots - a_0 s^{d-1}\right).$$

Thus $s \mid a_d r^d$, but since $gcd(r, s) = 1$ we have $s \nmid r^d$ and so $s \mid a_d$.

On the other hand, subtracting everything except $a_0 s^d$, we have

$$-a_d r^d - a_{d-1} r^{d-1} s - \cdots - a_1 r s^{d-1} = a_0 s^d.$$

Factoring out an r, we have

$$r\left(-a_d r^{d-1} - a_{d-1} r^{d-2} s - \cdots - a_1 s^{d-1}\right) = a_0 s^d$$

so that $r \mid a_0 s^d$. Since $gcd(r, s) = 1$, we have $r \nmid s^d$ so $r \mid a_0$. □

Example: If a high school student were told to find rational roots of $x^3 + x^2 - 5x - 2$ they would know to try $x = \pm 1$ and $x = \pm 2$. However if they were given the polynomial $3x^3 + 7x^2 - 3x - 2$ they probably would still try $x = \pm 1, x = \pm 2$. However, they probably wouldn't think to attempt $x = \pm\frac{1}{3}$ or $x = \pm\frac{2}{3}$.

Remember the following:

Theorem (The Fundamental Theorem of Algebra)**.** *A polynomial of degree d can have at most d roots.*

If we were only concerned with a polynomial with with rational coefficients than we can use the rational root theorem repeatedly to show that it can have no more than d rational roots.

To prove the fundamental theorem of algebra, we could suppose, for contradiction, that the polynomial has more than d roots. Now multiply out to show that we would end up with a polynomial of degree greater than d.

Parallel: integers and polynomials lead to rationals and rational funtions

What is a rational function? It is a funtion which is the ratio of two polynomials, ie $r(x)$ is a rational funtion if $r(x) = \frac{p(x)}{q(x)}$ where $p(x), r(x)$ are polynomials.

We need to have some equivalence between polynomials:

When is $\dfrac{p(x)}{q(x)}$ equivalent to $\dfrac{r(x)}{s(x)}$? This is similar to the question of when to rational numbers are equivalent and the answer is very much the same. They are equivalent when $p(x)s(x) = r(x)q(x)$ but only for all x where both rational funtions are defined.

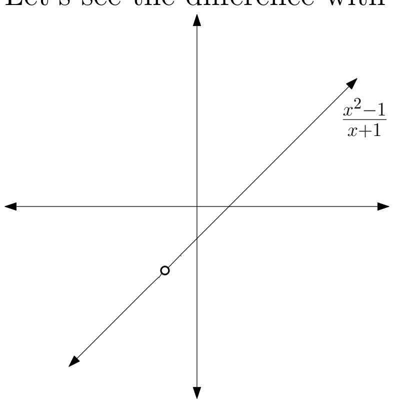
Example: Is it true that $\dfrac{(3x+2)(x-1)}{(4x-5)(x-1)}$ is equal to $\dfrac{3x+2}{4x-5}$?

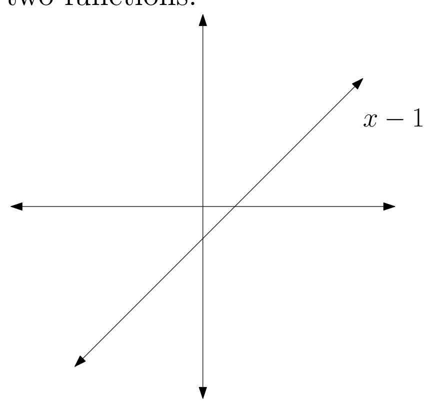
This is only true where they are both defined: it is *not* true for $x = 1$ or $x = \frac{5}{4}$.

High school students often don't understand this and they often don't understand why we hold so much importance on the domain of a function.

Simplify: $\dfrac{x^2-1}{x+1} = \dfrac{(x+1)(x-1)}{(x+1)} = x - 1$ *only if* $x \neq -1$.

Let's see the difference with graphs of the two functions.





Partial Fraction Decomposition

Suppose we had the fraction $\frac{3}{x^2-1}$ and we wanted to express it as the sum of two fractions. How do we do this?

Set up the equation: $\dfrac{3}{x^2-1} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$. From this point we can do one of two things:

1. Multiply through by $x^2 - 1$ to get the equation $3 = A(x-1) + B(x+1)$ then, plugging in $x = \pm 1$ we can solve for A and B.

2. Group like terms together, that is $3 = (A+B)x + (-A+B)$ and solve the equations $A + B = 0$ and $-A + B = 3$.

The two methods are essentially equivalent.

Partial fraction decomposition is not always taught in the high school curriculum, although it used to be. We need this technique in integral calculus (for example, to calculate $\int \frac{dx}{x^2-1}$) and differential equations, but it is also useful in other applications.