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MAE301 class notes

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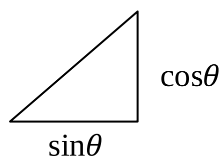
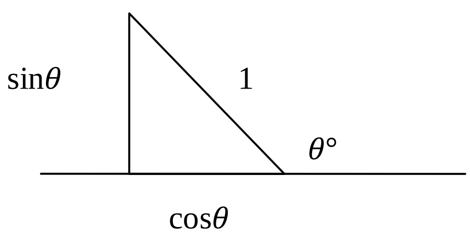
We have defined the trigonometric functions as ratios of the side lengths of right triangles. Since they are ratios, we may choose a representative triangle, with hypotenuse of length1; then one leg is sinθ and the other leg is cosθ.



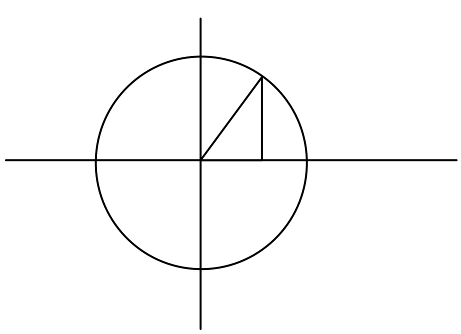
In this setup, $0° < \theta < 90°$. In order to generalize to $\theta > 90°$, we'd have



We want $cos\theta < 0$ because the base of the right triangle is "behind" the actual length… we have to back up. This is a reasonable way to extend this notion geometrically. Of course, this is only a heuristic discussion. Similarly, if the angle θ is between $180°$ and $270°$, the two legs would have a negative length (so both the cosine and sine would be negative), and in the fourth quadrant we would have the cosine positive again, but the sine would be negative.

By continuity, when the angle is right, we have a "triangle" with a leg of length 1, the base of length 0, and the hypotenuse of length 1 (as usual). But these aren't triangles, and this isn't a "real" definition.

For a more sound foundation, we transfer the discussion to the unit circle.

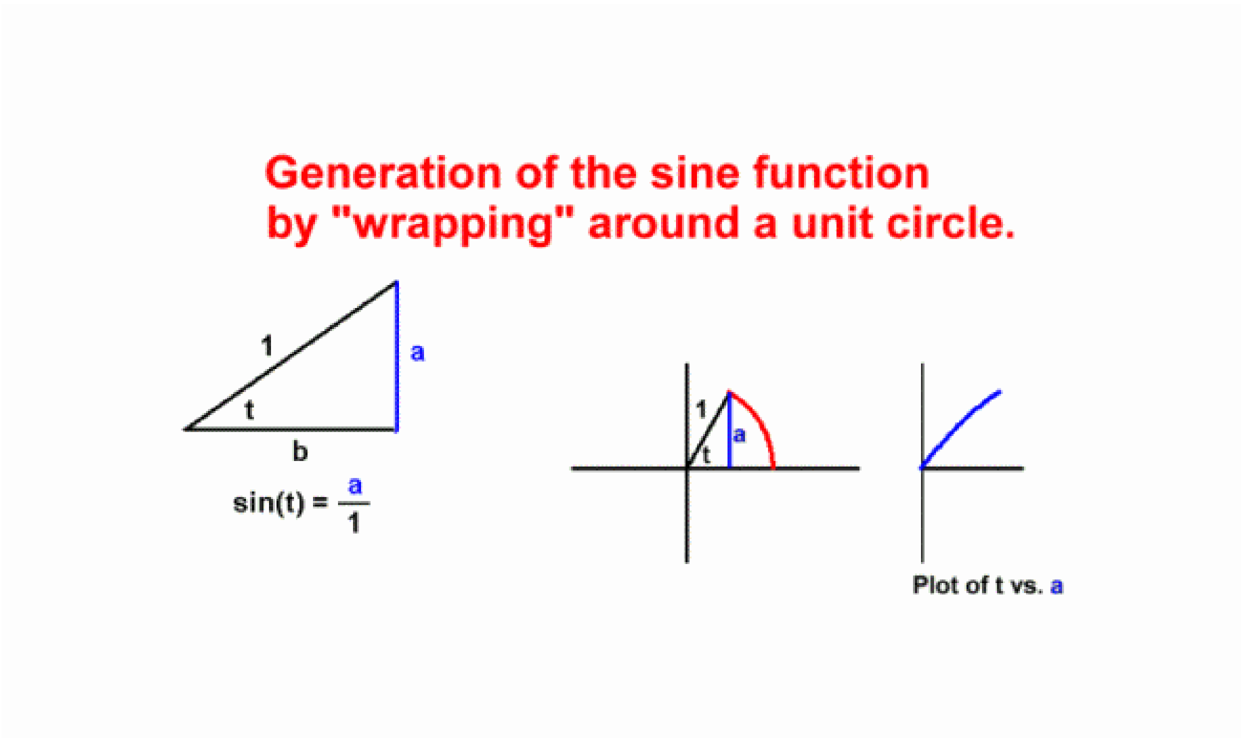


y = sinθ
x = cosθ
$\sin^2\theta + \cos^2\theta = 1$

Let x and y be points on the unit circle, and let $\sin\theta = y$ and $\cos\theta = x$. θ is the angle that the line from (0,0) to (x,y) makes with the x axis. (At this point, it is easy for students to get lost in the switch from heuristic to 'sound' discussion. As a teacher, you need to make it clear that we are extending the definition from ratios of sides to another one which agrees when the angle is acute. It is very hard for students to switch from a restrictive definition to a more general one, and you need to work hard to help students understand what is going on.)

Also, instead of degrees, we may use radians to measure angles. Radians are naturally defined, in that they relate angle measurement to a length of an arc. Degrees, on the other hand, are a completely arbitrary (albeit quite familiar) system of measurement. Another "natural" measure is "turns": one full rotation counter-clockwise as 1 turn

High school students learn how to change an angle measure from degrees to radians, usually in pre-calculus.



See an animated version on the web, or also this animation.

This should also make it clear how a $\cos\theta$ graph should look.

Now we can "forget" triangles if we want, and we have a function $\sin: \mathbb{R} \to [-1,1]$, and think in terms of its graph. The problem here is making the 'cognitive shift'… i.e. it's easy to lose students.

Notice that polynomials, rational functions, logs, exponentials are unbounded. Sinθ and cosθ are bounded functions, with [-1,1] as the range. They are also periodic functions; a very different kind of function from polynomials and exponentials.

Of course, there is a deep relationship between the exponential, the sine, and the cosine, via Euler's formula: $e^{i\theta} = cos\theta + isin\theta$. Usually in high school we don't go into this relationship.

Note that this gives us $e^{i\pi}$ = -1.

This relationship is useful to figure out sin2θ in terms of sinθ and cosθ. We could also do this with triangles, of course.

$$e^{i(2\theta)} = cos2\theta + isin2\theta$$

$$= (e^{i\theta})^2 = (cos\theta + isin\theta)^2$$

$$= cos^2\theta + 2isin\theta cos\theta + i^2sin^2\theta$$

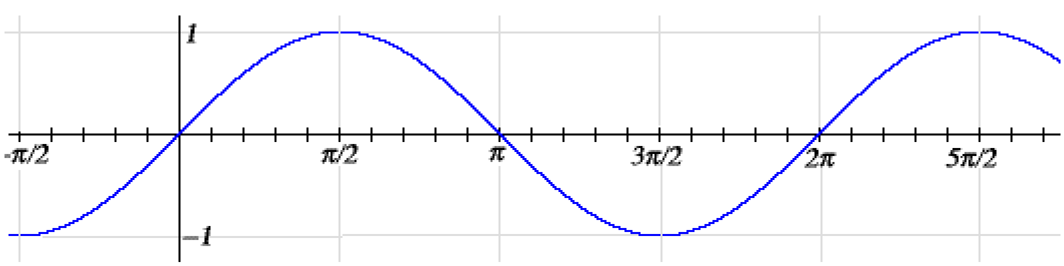
$$= (cos^2\theta - sin^2\theta) + i(2sin\theta cos\theta)$$

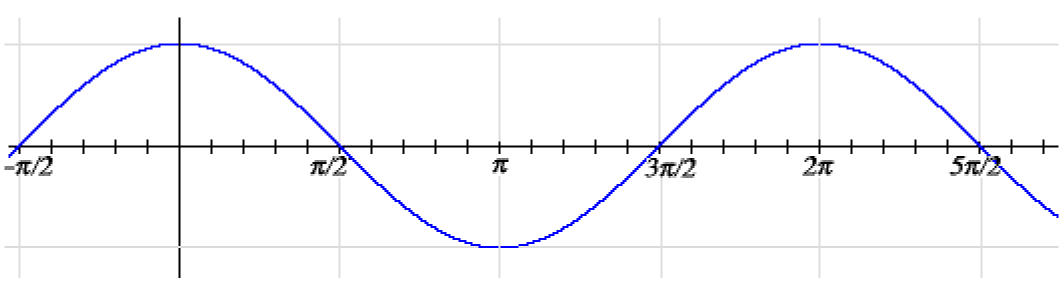
$$= cos2\theta = cos^2\theta - sin^2\theta$$

$$sin2\theta = 2sin\theta cos\theta$$

We can figure out the graph of tanθ from that of sinθ and cosθ: since

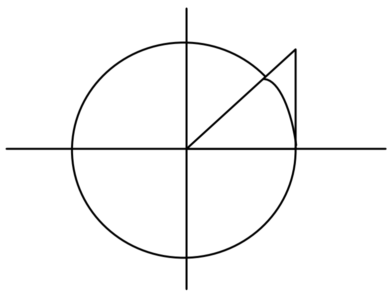
tanθ = $\frac{sin\theta}{cos\theta}$, we note that the tangent is 0 whenever the sine is 0, and we get an asymptote when the cosine is 0.





Furthermore, we can figure out that the tangent is always increasing by looking at the relationship between the sine and the cosine graphs.

We can also produce the graph of tanθ in the same way we did for the sine, on the unit circle. For angles between -90 and 90, the tangent is the vertical leg of the right triangle whose base is 1. As the angle increases from 0, the height grows towards infinity. Similarly, for negative angles, we see that the height of the triangle is below the axis.



$$\frac{sin\theta}{cos\theta} = \frac{tan\theta}{1}$$