Relations: What are some of the relations of interest to us?

- functions
- equivalence relations (which lead to partitions)
- order

If you have an equivalence relation on \( \mathbb{X} \) then you can form a partition of \( \mathbb{X} \) into equivalence classes.

\[
[a] = \{x \in \mathbb{X} | x \sim a\}
\]

\( \mathbb{X}/\sim = \{[a] | a \in \mathbb{X}\} \)

This is what we’ve already been doing to build up to the set of Reals from the Naturals:

\[
\mathbb{N} / \sim_1 = \mathbb{Z}
\]

\[
\mathbb{Z} / \sim_2 = \mathbb{Q}
\]

\[
\left( \prod_{i=1}^{\infty} \mathbb{Z}_{10} \right) / \sim_3 = \left( \prod_{i=1}^{\infty} \mathbb{Q} \right) / \sim_4 = \mathbb{R}
\]

Functions:

- In elementary school there are numbers and operations (using numbers to get more numbers).
  - Any symbols are like constants (such as the area of a square: \( A = s^2 \))
- In Jr. High to (early) High school variables come into play.
  - Now symbols can be variables or constants (ex: \( y = 3x + 2 \)). However there is an issue of how to calculate vs. a relation.
In senior high school through college (starting roughly with precalculus) functions become mappings.

*Even though there is this increase in abstraction we must keep in mind that some students at the higher levels might still be thinking in the earlier ways.*

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When is a function $f : A \rightarrow B$ invertible?

(Note that this is mildly different from: “given a function $f(x) = \frac{3x+2}{5x-4}$, is it invertible?”)

$f$ is invertible if it is one to one. (It doesn’t necessarily matter if it is onto since we can restrict the domain of $f^{-1}$ to be the image of $f$. \{ $y | f(x) = y$ for $x \in \text{dom}(f)$ \})

So given $f(x) = \frac{3x+2}{5x-4}$

Find $f^{-1}(x)$

Process: let $y = f(x)$, switch $x$ and $y$,

$x = f(y)$

Solve for $y$.

Graphically, the inverse is a reflection in the line $y = x$.

Students need to understand that the step to switch $x$ and $y$, $x = f(y)$, is purely notational.
Students will complain that this is wrong because you didn’t switch $x$ & $y$ first.

Since they don’t fully understand the concepts, they just want to follow steps.

Similarly, for $y = x^2$, without really saying it we restrict $f(x)$ to where it is one to one and solve.

$$x = y^2$$

<table>
<thead>
<tr>
<th>A:</th>
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<tbody>
<tr>
<td>Think: $y &gt; 0$ AND $x &gt; 0$</td>
</tr>
<tr>
<td>$\sqrt{x} = \sqrt{y^2}$</td>
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<tr>
<td>$y = \sqrt{x}$</td>
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<table>
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<tr>
<th>B:</th>
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<tbody>
<tr>
<td>Also: $x &gt; 0$ AND $y &lt; 0$</td>
</tr>
<tr>
<td>$\sqrt{x} = \sqrt{y^2} = -y$</td>
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<tr>
<td>$y = -\sqrt{x}$</td>
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<tr>
<th>A and B unified:</th>
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<td>$y = \pm \sqrt{x}$</td>
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*This is not a function. It’s shorthand for two functions.*

Ideally $\pm$ would not be used. It would be written as two parts so students understand you’re considering two possibilities.

Generation definition of a function:

A=Domain={1,2,3}

B=Co-Domain={4,5,6}

How many functions are there from $A \to B$?

The answer is $3 \times 3 \times 3 = 27$

What about invertible functions?

$f(1) =$Choose 1 of the 3 in B$}

$f(2) =$Choose 1 of the remaining 2 in B$}

$f(3) =$Choose the last one in B$}
This is $3 \times 2 \times 1 = 6$ invertible functions.

What about for $f: \{1,2,3\} \rightarrow \{4,5,6,7\}$

There are $4^3$ functions with $4 \times 3 \times 2$ invertible.

**A function is completely determined by its input and output**

The question of functions can be recast in the following way:

“How many permutations are there of $\{4,5,6,7\}$ taking 3 at a time?”

Rephrase and ask ourselves:

“How many ways can we choose 3 things out of a set of 4?”

\[
\binom{4}{3}
\]

This is the number of permutations of 3 out of 4 divided by the number of permutations of those 3 things.

In general:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

Where $\frac{n!}{(n-k)!}$ is the permutation part

And $\frac{1}{k!}$ is the arrangement part

In Summary:

- Ordering $n$ objects out of $k$ with repetition allowed is $n^k$
- Ordering $n$ objects out of $k$ without repetition is $\frac{k!}{(k-n)!} \cdot (\text{Permutation } kP_n)$
- Choose $n$ out of $k$ objects without repetition or order is $\frac{n!}{k!(n-k)!} \cdot (\text{Combination } \binom{n}{k})$