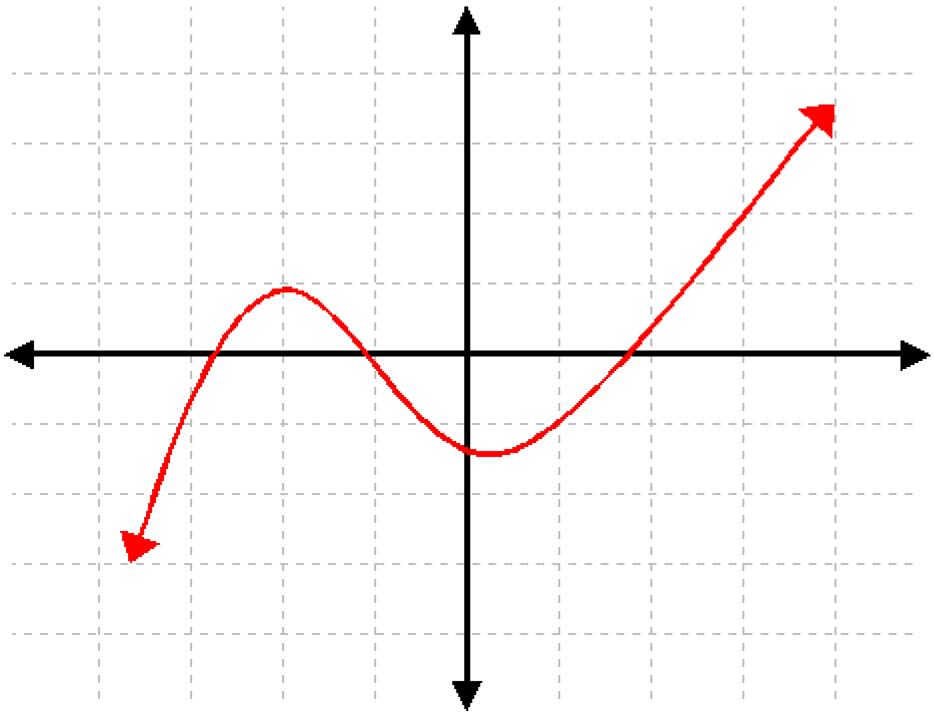
Functions

Here are some ways to give a function:

- formula or table of values
- a plot or graph
- an algorithm that computes it
- description of its properties

Here is a graph that represents a function.

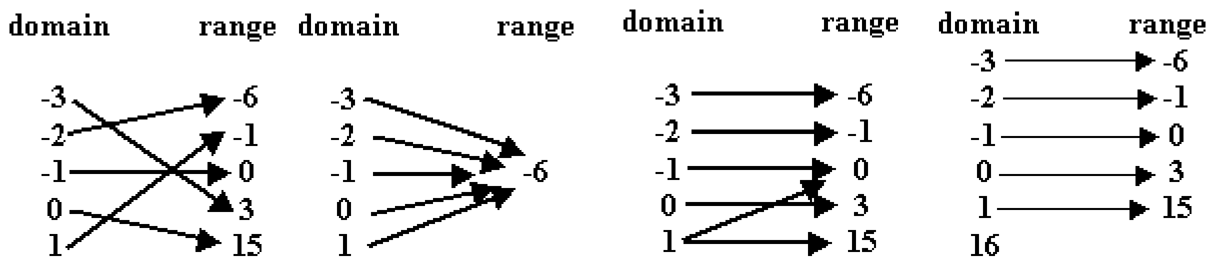


Notation:

$f(\mathrm{x})\text{: A}\rightarrow\text{B}$

Here the A is the set of all the inputs, and B is the set of all the outputs. The proper name for A is the domain of the function and B is the codomain, also known as the range. The domain is all the *x*-values and the codomain is all the *y*-values. Each element of the domain maps to an element in the codomain. In order for us to say that $f(x)$: A→B is a function we might say that the graph passes the vertical line test. More formally, without using the notion of a graph we can say that every *x* in the domain is assigned to only one *y* in the codomain.

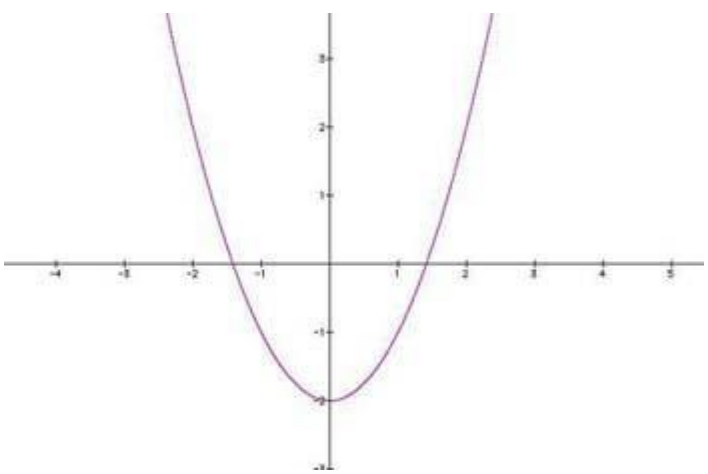
Which of the following represent a function?



This is a function. This is a function. This is not a function. This is not a function.

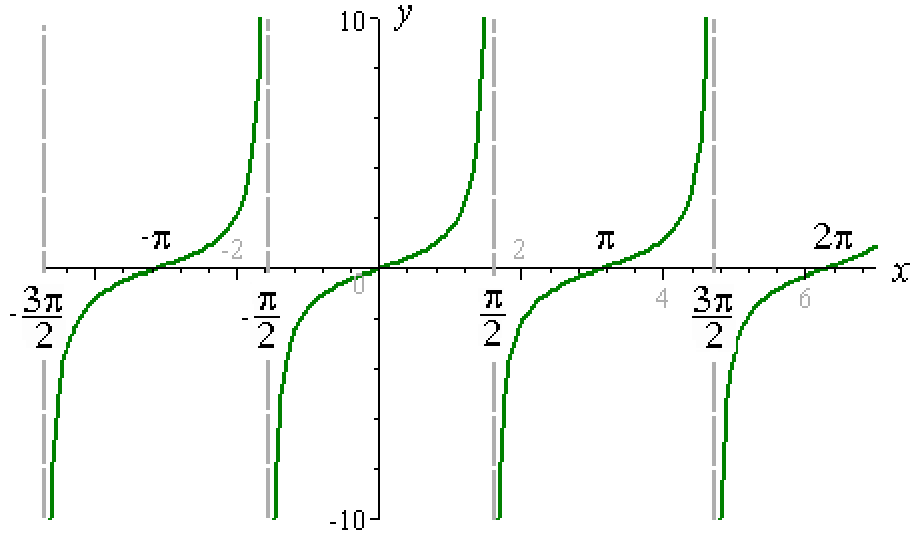
The first one is a function because each element in the domain assigns to an element in the range; similarly, the second one is a function. The third one on the other hand is not a function because 1 from the domain is assigned to two different elements in the range. The last one is not a function as well because 16 in the domain is not assigned to any element in the range.

We talked about that if the vertical line passes the graph once then the graph is a function. Does the vertical line intersect the graph at most or exactly once?
The answer to this is that it depends on the domain of the function. Let's consider the two possibilities.
The following graph represents a function with the domain being $(-\infty, \infty)$



Looking at the graph the vertical lines intersects the graph exactly once. If we were to restrict our domain to $\{x \in \mathbb{R} | x \geq 0\}$ then we would say that the vertical lines intersect the graph at most once, because there would be no intersection for all the negative *x*-values.

Another example would be if we look at the graph *y*=*tan*(*x*). Strictly speaking, this is not a function whose domain is the reals, since at points such as $\frac{\pi}{2}$,-$\frac{\pi}{2}$, $\frac{3\pi}{2}$,-$\frac{3\pi}{2}$, etc, *y*=*tan*(*x*) is not defined at those values. However, it is function at all other values; the domain is implied to be all reals except for odd multiples of $\frac{\pi}{2}$.



So far, all the functions discussed were mathematical functions, but here is an example of a perfectly good function:

f: students at stony brook university→ID numbers

Here we have that each student is assigned an ID number, and there is no student who is assigned to more than one ID number. This clearly represents a function, and is a good way to start of introducing the meaning of functions to students. Students are often told that a function is some calculator button, meaning that you put in a number *x* and out comes the *y.* The trouble with this is that sometimes one must press several buttons, or enter several numbers, and some buttons (such as the off button) don't give a result at all. So this metaphor can be misleading. Another example might be the dating and cheating model, where we say that *x* dates *y* and only one y because if *x* dated more than one *y* then *x* would be cheating and we know that cheating is bad. These examples are not functions, but rather models to understand functions and they are good as long as they work.

More formally, Given sets A, B (A-domain, B-codomain)
AxB= {(*a*,*b*)I *a*∈A, *b*∈B}
A function is a subset of AxB such that each element *a* has a unique *b* in the pair (*a*,*b*).
b=*f*(*a*) paring (*a*,*f*(*a*)), where *a* is the input and *f*(*a*) is the output.

A function *f* is invertible if the paring is such that for each *b*∈B there is a unique *a*∈A, so that (*a*,*b*)∈*f*

More generally, any subset of AxB is called a relation. A function is just a "special" relation, or sometimes referred to as "well-behaved" relation. When we say that a function is "a well-behaved relation", we mean that, given a starting point, we know exactly where to go; given an *x*, we get only one *y*. Keep in mind that since all functions are relations, not all relations are functions. Functions are sub-classification of relations.

Let’s consider another kind of relations called an equivalence relation.
An equivalence relation “acts like =”

An equivalence relation is a binary relation on a set that specifies how to partition the set into subsets such that every element of the large set is in exactly one of the subsets.

Any two elements of the larger set are then considered ‘equivalent” with respect to the equivalence relation if and only if they are also elements of the same subset.

Notation:

If *a,b* are elements of a set and are equivalent with respect to equivalence relation *R* then we can write that *a* is related to *b* in the following ways.

- $a \sim b$
- $a \equiv b$
- $a \sim_R b$
- $a \equiv_R b$
- aRb

Let *A* be a set and ~ be a binary relation on *A,* then ~ is called an equivalence relation if and only if for $a, b, c \in A$ all the following are true:

- Reflectivity: $a \sim a$
- Symmetry: if $a \sim b$ then $b \sim a$
- Transitivity: if $a \sim b$ and $b \sim c$ then $a \sim c$

Example:

Two rationals $\frac{p}{g}$ and $\frac{r}{s}$ are related if $ps = rq$

$$\frac{-6}{-12} \sim \frac{13}{12} \sim \frac{1}{2}$$

All these fractions represent $\frac{1}{2}$

“Equivalence classes”

Given a set *A* and an equivalence relation ~ on *A*, the equivalence class of an element $a \in A$ is the subset of the elements in *A* which are equivalent to a This is typically denoted by $[a]$

$[a] = \{x \in A \mid x \sim a\}$

For example

Let's start with ℤx(ℤ/ {0}) where $p, q \in \mathbb{Z}$

$(p, q) \sim (r, s) \Leftrightarrow ps = rq$

ℤx(ℤ/{0})/~

$\mathbb{Q} = \{(p, q) | \ p, q \ are \ in \ least \ terms\} \ (p, q) = 1 \ q > o$

$\mathbb{R} = \{Cauchy \ Sequence\}/\sim$

$\{a_i\} \sim \{b_i\}$ if *lim* a_i = *lim* b_i

<u>Strict ordering</u>

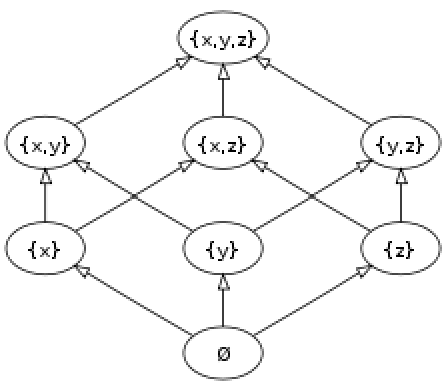
a>b

Irreflexive (that is, reflexivity never holds) and asymmetric (for all a, b, if a<b, then b<a is fals) but it is transitive because if *a>b* and *b>c* ⇒ *a>c.*

<u>Non-strict total ordering</u>

Reflexive, transitive, antisymmetric *a≥b* and *b≥a* ⇒ *a=b.*

<u>Partial ordering</u>



$$\emptyset < \{x\} < \{x, y\} < \{x, y, z\}$$

$$\emptyset < \{x\} < \{x, z\} < \{x, y, z\}$$

$$\emptyset < \{y\} < \{x, y\} < \{x, y, z\}$$

$$\emptyset < \{y\} < \{y, z\} < \{x, y, z\}$$

$$\emptyset < \{z\} < \{y, z\} < \{x, y, z\}$$

$$\emptyset < \{z\} < \{x, z\} < \{x, y, z\}$$

Notice that not all pairs can be defined. In a partial ordering, you cannot always compare all the elements to each other.