MAE 501 - Note for 03/10/09

Discussion of some exam problems

(21) Determine the distance between points A (-1,-3) and B (5, 5). Write an equation of the perpendicular bisector of $\overline{\mathbf{AB}}$.

Solution:

$$\text{Distance of } \overline{\mathrm{AB}} = \sqrt{(-1-5)^2 + (-3-5)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

$$\text{Slope of } \overline{\mathrm{AB}} = \frac{5-(-3)}{5-(-1)} = \frac{8}{6} = \frac{4}{3}$$

If two lines are perpendicular to each other, then the product of the slopes of these two lines is equal to -1.

$$\text{Slope of the perepndicualr bisector of } \overline{\mathrm{AB}} = \frac{-1}{\text{Slope of } \overline{\mathrm{AB}}} = \frac{-1}{\frac{4}{3}} = -\frac{3}{4}$$

$$\text{Midpoint of } \overline{\mathrm{AB}} = \left(\frac{5+(-1)}{2}, \frac{5+(-3)}{2}\right) = (2, 1)$$

Thus, $\frac{y-1}{x-2} = -\frac{3}{4} \quad \Rightarrow \quad 4(y-1) = -3(x-2) \quad \Rightarrow \quad 4y - 4 = -3x + 6 \quad \Rightarrow \quad 3x + 4y - 10 = 0$

The equation of the perpendicular bisector of $\overline{\mathrm{AB}}$ is $3x + 4y - 10 = 0$.

Note:

To deal with these kind of problems, it is most appropriate to use either the two-point form or the point-slope form of the equation of a line. However, most high school students or college freshmen would only remember the equation of a line is

$$y = mx + b$$

However, many of them actually don't know how to use the equation to find out the actual equation when points are given, because finding the intercept (b) requires some understanding of what the equation of the line actually means, rather than just manipulating some symbols.

Since we know the slope of the perpendicular bisector is -3/4, we solve this problem by substituting (-3, -1) into the equation we have: $3 = \left(-\frac{3}{4}\right)(4) + b$. Then, we solve for b and find out the equation is $-\frac{4}{3}(x-4) = y - 3$.

Rather than memorize a specific formula, it is better to encourage your students to understand the concept. First we recall that the slope of a line is the ratio of the change in y (the Rise) over the change in x (the Run)

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{y_1 - y_0}{x_1 - x_0} \qquad (1)$$

Since we can use any two distinct points on a the line, we could replace x_1 with x and y_1 with y. Then clearing the denominator gives the point-slope form of the line

$$y - y_0 = m(x - x_0) \qquad (2)$$

Substitute (1) into (2), we have the two-point form:

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

(15) The equation $\mathbf{W = 120I - 12I^2}$ represents the power (W), in watts, of a 120-volt circuit having a resistance of 12 ohms when a current (I) is flowing through the circuit. What is the maximum power, in watts, that can be delivered in this circuit?

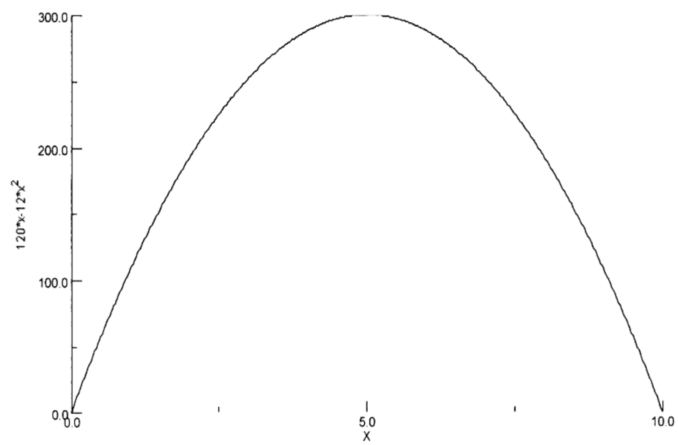
<u>Method I: Use calculus</u>

$$\frac{dW}{dI} = 0 \quad \Rightarrow \quad 120 - 24I = 0 \quad \Rightarrow \quad I = 5$$

I=5 is the critical point. Hence, the maximum power can be delivered in this circuit is

$$120(5) - 12(5^2) = 300 \text{ watts}$$

This method is not expected, because most of the high school students do not know calculus.



<u>Method II: Use the property of parabola</u>

When W=0, $120I - 12I^2 = 0$. Then, we have $I = 0$ or $I = 10$. Recall the vertex of parabola is at the middle between $I = 0$ or $I = 10$. Therefore, when $I = \frac{0+10}{2}$, then vertex of the parabola is at $I = 5$. (Some students would remember that the vertex of the general quadratic $ax^2 + bx + c$ occurs at $x = -\frac{b}{2a}$ instead.) Therefore, the maximum power can be delivered in this circuit is

$$120(5) - 12(5^2) = 300 \text{ watts}$$

<u>Method III: By completing the square</u>

$$120I - 12I^2 = -12(I^2 - 10I) = -12(I^2 - 10I + 5^2 - 25) = -12(I - 5)^2 + 300$$

The minimum of $-12(I-5)^2$ is equal to zero when $I = 5$. Hence, when $I = 5$, $-12(I-5)^2 + 300$ is at its maximum, which is equal to 300. Therefore, the maximum power can be delivered in this circuit is

$$120(5) - 12(5^2) = 300 \text{ watts}$$

Now, back to our regularly scheduled program already in progress:

<u>"Laws" of Exponents</u>

$$x^2 * x^3 = x^{2+3} = x^5$$
$$(x^2)^3 = (x * x)(x * x * x) = x^5$$

Students can absorb these easily, and are usually quite comfortable with them. However, when students need to deal with the transfer from whole numbers to integers or rational numbers as exponents, they will often make what seem to be very silly mistakes. For example, we know $x^{-3} = \frac{1}{x^3}$ and $x^{2/3} = \sqrt[3]{x^2}$, but some students will do these incorrectly such as $x^{-3} = \sqrt[3]{x}$ or $x^{1/3} = \frac{1}{x^3}$. Students do these wrong, because they just memorize how the outcome looks like, they actually do not know the concepts behind the problems. Consequently, they mix up whether a fractional exponent should be a fraction or a root.

Exponent Law	Rule
Multiplication	$x^a * x^b = x^{a+b}$
Quotient	$x^a / x^b = x^{a-b}$
Power of a Power	$(x^a)^b = x^{a*b}$
Zero Power	$x^0 = 1, provided\ x \neq 0$
Negative number	$x^{-a} = \frac{1}{a^x}\ and\ \frac{1}{a^{-x}} = a^x, provided\ x \neq 0$
Fractional exponent	$x^{p/q} = \sqrt[q]{x^p}$

These rules are not just arbitrarily assigned: they follow directly from the properties of the real numbers.
The multiplication rule is really just codifying the fact that for powers which are natural numbers, exponentiation is just repeated multiplication. For example,

$$x^5 = x * x * x * x * x = (x * x * x) * (x * x) = x^3 * x^2 = x^{2+3}$$

From this, the quotient rule follows immediately, at least when $a > b$, since $x^{a-b} * x^b = x^a$. Then dividing both sides by x^b gives us $\frac{x^a}{x^b} = x^{a-b}$. We now extend this rule to hold even when $b = a$ (giving us $x^0 = 1$) and $b > a$, which forces the definition $x^{-a} = \frac{1}{x^a}$ (since $x^{-a} = x^{0-a} = \frac{1}{x^a}$).
Similarly, the definition $(x^a)^b = x^{ab}$ follows for whole number values of b from the multiplication rule (for example, $(x^2)^3 = (x^2) * (x^2) * (x^2) = x^{2+2+2} = x^6$. We then extend this to arbitrary powers, which forces the definition for fractional exponents on us, as we can see below.

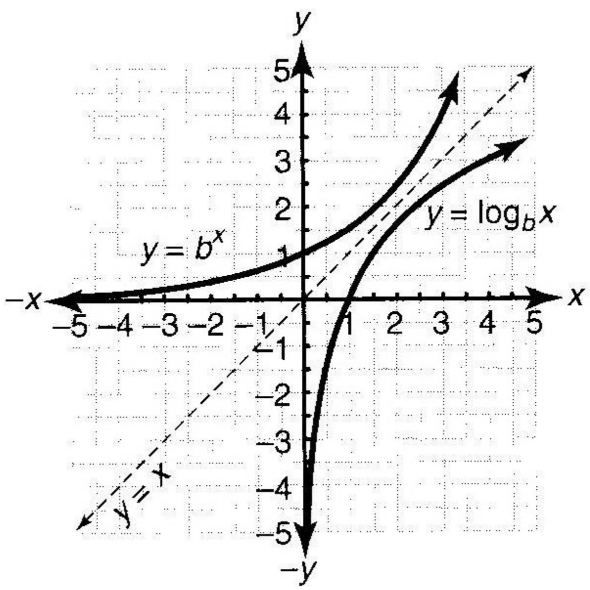
The square root of a number x is one of two identical numbers whose product is x. The symbol $\sqrt{\ }$ is called a radical sign, and the number underneath the radical sign is the radicand. Similarly, the cube root of a number x is one of three identical numbers whose product is x, and is denoted as $\sqrt[3]{x}$. In general, $\sqrt[k]{x}$ represents k identical whose product is x, where k is the index of the radical. By extending the laws of exponents to include rational exponents, we know $x^{1/k}$ means $\sqrt[k]{x}$, where k indicates what root of x is to be taken. Therefore, we can write

$$x^{1/k} * x^{1/k} * x^{1/k} * x^{1/k} * \ldots * x^{1/k} = x$$

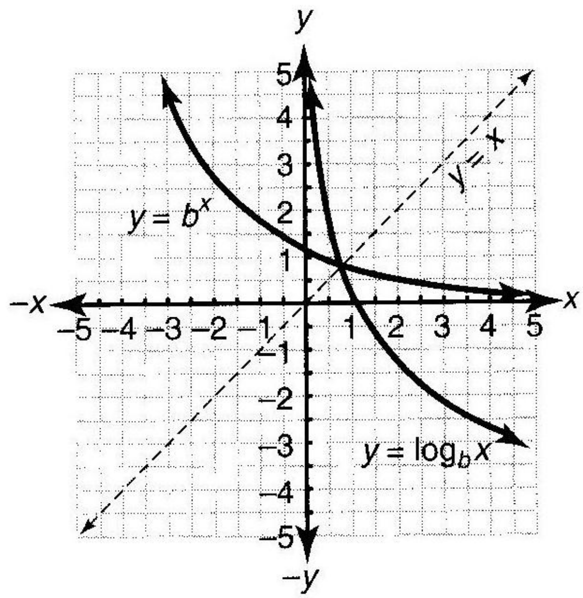
The expression $\sqrt[k]{x}$ may represent a real number or an imaginary number. If x is an odd number, then $\sqrt[k]{x}$ is always a real number. If x is an odd number, then $\sqrt[k]{x}$ is always an imaginary number. With regarding to the laws of exponent and using exponents to indicate roots, we can evaluate $x^{n/k}$ into two ways. The first one is to find the kth root of x and then raise the result to the nth power such that $x^{n/k} = (x^{1/k})^n$; the second way to raise x to the nth power and then take the kth root of the result, we have $x^{n/k} = (x^n)^{1/k}$. As teacher, we may have to translate $x^{1/k} = \sqrt[k]{x}$ to $8^{1/3} = \sqrt[3]{2} = 2$, because for many students these don't seem to be the same.

The exponential function and logarithmic function

The equation $y = b^x$ is an exponential function, where our variable is in the exponent, provided that b is a positive number other than 1. The inverse of the exponential function $y = b^x$ is $x = b^y$. There is no algebraic method for solving $x = b^y$ for y in terms of x that works for all values of y. Therefore, logarithmic function is introduced to allow y to be expressed in term of x. The logarithm of x to the base b is expressed $log_b(x)$. So, for a number x, a base b and an exponent y, we can write $y = log_b(x)$ if and only if $x = b^y$. The graph of exponential function $y = b^x$ contain (0,1) and has no x-intercept, the graph of logarithmic function of the form $y = log_b(x)$ contains (1,0) and has no y-intercept.



Graph of $y = b^x$ and $y = log_b(x)$ with b>1

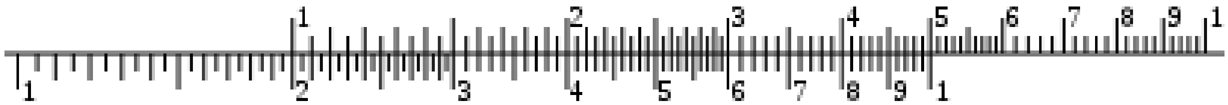


Graph of $y = b^x$ and $y = log_b(x)$ with b>1

Let us look at some values of the function $f(x) = 2^x$.

X	$f(x) = 2^x$
-2	¼
-1	½
0	1
1	2
2	4
3	8
4	16

We can use two number lines (or rulers) to add numbers by adding lengths. For example, to add 3 + 5, we position the top ruler so that it starts at the 3 of the bottom ruler. Then we look at where the 5 on the top ruler falls on the bottom ruler (in this case, the 8). We could also use a similar method to subtact (to get 6-2, but the 2 over the 6, and look where top ruler begins). We can construct a pair of rulers for multiplying in a similar way. Instead of marking off the ruler so that equal distances denote equal numbers, we mark the rulers off with a logarithmic scale: the distance from 1 to 2 should be the same as from 2 to 4, and from 4 to 8, etc. We can fill in the other numbers, as well. Then, to multiply two numbers, say $x * y$, we put the 1 of the top ruler over the x and then see what number on the bottom ruler the y falls over. This is exactly the principle that a slide rule is built on.



Division can be done in a similar way.

This leads us to the rules for logarithms:

$$x^a * x^b = x^{a+b}$$
$$log_x\left(x^a * x^b\right) = log_x x^a + log_x x^b = a + b$$

So, if we fix the base, we can add logs

$$\log(zw) = logz + logw$$
$$x^{-n} = \frac{1}{x^n} \quad \Rightarrow \ -\log(n) = \log(1/n)$$

Reference

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