Michael Cento – Notes from February 26

<u>Recall</u>: Going from **Q** to **R** by simple "throwing in" the irrational numbers is unsatisfying since "irrational" cannot be defined in this way.

High school students naturally make a transition from **Q** to the Algebraic numbers, with a few additional elements such as π and e that are not algebraic numbers. (These are called transcendental numbers).

Algebraic Numbers: Real (and ONLY Real) solutions of polynomials.

<u>Theorem</u>: If p(x) is a polynomial of degree ≤ 4, then the solutions can be expressed in terms of radicals (i.e. $\sqrt{2}$, $\sqrt{3}$, …)

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \ , \ a_i \in \mathbf{Z}$$

BUT, there exist polynomials of degree ≥ 5 for which this is not true (i.e. there is no formula, and the solutions cannot be).

For example: $x^5 = 32$. This is easy to solve, since the 5^{th} root of 32 is 2. However, there is no concrete formula that can be used to solve an equation such as

$$x^5 - 3x^4 - 11x^3 + 27x^2 + 10x - 23 = 0,$$

even though it isn't too hard to determine that the solutions are approximately -3.004, -0.9832, 0.9593, 2.032, and 3.995.

A high school student has a primitive understanding of sets of numbers such as the real numbers. A teacher can tell a student there are "other" numbers that exist in addition to the ones they already know of (such as integers, fractions, roots, etc) and most likely the student will be fine with this limited knowledge. We as teachers must be cognizant of this lack of knowledge on the part of the student.

Recall that when we first discussed real numbers earlier (2/5), we gave a definition of a real number as the set of all Cauchy sequences, where two sequences were identified if they had the same limit. Unfortunately, this definition is not one that most students can understand or identify with.

We also can formally define reals as Dedekind cuts (which we also discussed), but again, this is a difficult concept. Recall that a Dedekind cut is just a partitioning of the rational numbers into two sets, such that every number in one set (the "left" set) is less than every number in the other set (the "right" set). There 2 cases when making the Dedekind cut:

1) The cut is made at a rational number, so when the sea of numbers is divided into 2 sets, there is a either a largest rational element in the left set, or a smallest rational element in the right set. (Obviously, both cannot be true simultaneously, since between any two rationals there is another.)

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<----------](-----------> or <----------)[----------->
 Largest rational^                       ^Smallest rational
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2) The cut is made between rationals so there is no largest or smallest elements a in either set.

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Case (1) gives us back something that corresponds to the rational numbers, but case (2) gives something new (an irrational number).

Rather than going this route, we can define a real number to be any infinite decimal. Note that we can represent a "terminating decimal" like 3.4 by adding infinitely many zeros on the end.

Examples: 1.1427658319… or 1.0000000

As long as we agree not to use a representation that ends in all 9s, each such representation is unique.

Note that this is really nothing new. For example, we can view such infinite decimals as a way to identify a preferred Cauchy Sequence:

31796.81245 = 30,000 + 700 + 90 + 6 + .8 + .01 + .002 + .0004 + .00005

In general, we can represent any real number as

$$n + \sum_{i=0}^{\infty} a_i/10^i$$

where n is the integer part and a_i is the i^{th} digit to the right of the decimal point. The sequence of partial sums of the above is the preferred Cauchy sequence.

Here we agree that 0.999… = 1.000… An easy way to see this is to note that since

$$1/3 = 0.333...$$

multiplying both sides by 3 gives us

$$1 = 0.99\ldots$$

We have something similar to be true in any base:
Base 2: 0.111… = 1.0
Base 3: 0.222… = 1.0

Note also that an infinite decimal is really the same thing as a Dedekind cut. We are specifying the "right set" by giving a sequence of lower bounds on it (each time we write another decimal in the expansion, we move the bound to the right a little bit).

<u>Countable</u>: a set is countable if a natural number can be assigned to each member of the particular set (there exists a 1-1 correspondence with the natural numbers). An equivalent definition is that an ordering can be chosen so that each element in a set is a definite number of steps from the first element.
Q is a countable set, as are the algebraic numbers.

However, there are uncountably many real numbers, despite the fact that between any two real numbers, there lies another rational number.

Counting **Q**:

Q: {1/1, 1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, …}

1	2	3	4	5	6	7	8	…

<u>Solving Equations depends on the Domains of discussion</u>

N vs Z

$x + 3 = 5$ has a solution in **N**, namely $x = 2$

$x + 5 = 3$ has no solution in **N** (-2 is not a member of **N**)

Z vs Q

$3x = 6$ has a solution in **Z** , namely $x = 2$

$6x = 3$ has no solution in **Z** (1/2 is not an integer)

Q vs R or C

$x^2 = 4$ has solutions in **Q** , namely $x = 2$ and $x = -2$

$x^2 = 2$ has none since radicals are not part of **Q**

$x^2 = -1$ has none since numbers of the form $a + b\mathbf{i}$ (a, b in **R**) are not part of **Q**

In all of these cases, we have a desire to fill these gaps. (Of course, the jump from algebraic numbers to **R** is motivated by a desire for continuity rather than existence of solutions). Furthermore, historically the complex numbers are a useful computational tool, even if one doesn't want to admit them as "true" numbers.
For example, in order to solve the general cubic ($x^3 + ax^2 + bx + c = 0$), it is in fact simplest to compute with complex solutions, even to find the real (**R**) solutions.

Similarly, you may remember from differential equations that in solving a second order linear equation, you sometimes get solutions of the form $y = e^{(a + bi)x,}$ and then by various manipulations arrive at a purely real solution of the form $y = e^{ax}(c_1\cos(bx) + c_2\sin(bx))$

We let **i** be a solution $x^2 = -1$ (that is, $\mathbf{i} = \sqrt{-1}$). Now we set

$$\mathbf{C} = \mathbf{R}[i] = \{a + b\mathbf{i} \mid a,b \in \mathbf{R}\}$$

So to go from the Reals to the Complex numbers it is enough to adjoin the new number **i** alone. The following theorem guarantees this is enough.

<u>The Fundamental Theorem of Algebra</u>:

Let f(x) be a polynomial in **R** of degree n = 1, 2, 3 or 4. Then f(x) has exactly n roots in **C** (counting multiplicity). Equivalently, f(x) factors completely in **C**[x] into n linear factors. (Irving)

The Fundamental Theorem of Algebra has important consequences in high school mathematics. For example, we can deduce that the graph of a polynomial of degree d has at most d-1 turning points ("bumps").

References

Irving, R. Integers, Polynomials, and Rings. 2004, Springer-Verlag New York, Inc.