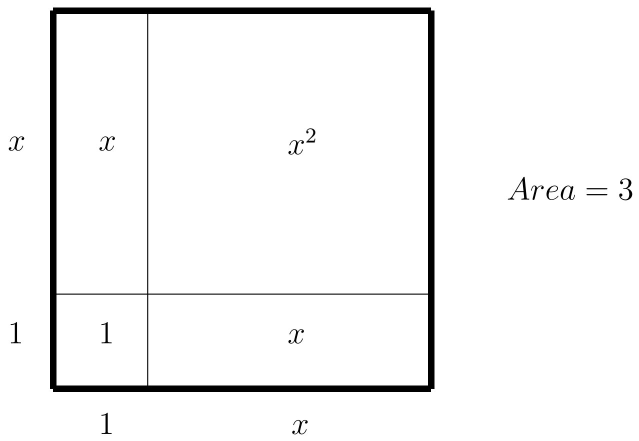
Intro/Example

As an introduction to the concept of completing the square and the quadratic equation, we were asked to consider a quadratic equation and how to convince a student that it had real roots without exhibiting what these roots are.

An example of this would be the equation, $x^2 + 2x - 1 = 0$

We could approach it graphically, ie: graphing the function $f(x) = x^2 + 2x - 1$, or we could evoke the Intermediate Value Theorem and argue that since $f(x)$ is continuous and $f(-3) > 0$, $f(-1) < 0$, and $f(1) > 0$ then our function must equal zero at least two real numbers (one between -3 and -1 and the other between -1 and 1).

But perhaps the clearest way to exhibit that the equation can be satisfied is to use a geometric argument. We can add 3 to both sides of the equation, yielding $x^2 + 2x + 1 = 3$ and notice that this is equivalent to $(x + 1)^2 = 3$. So we can think of this as a square of length $x + 1$ whose area is 3.



This is a believable explaination that our original equation has at least one real root. It basically utilizes the method of Completing the Square.

How to complete the square

Given the expression $x^2 + bx + c$ with $b, c \in \mathbf{R}$, we want to get it into the form of a square plus a number. So we can divide b by 2, square the result, and add it to the expression (at the same time subtracting it so that we can keep equality). Why do this? Because it works!

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$$

Let's work through an example of completing the square. Given $x^2 + 8x + 3$ we can divide 8 by 2 to get 4, square 4 to get 16, and add and subtract 16 to our expression, which will look like:

$$x^2 + 8x + 3 = x^2 + 8x + 16 + 3 - 16 = (x + 4)^2 + 3 - 16 = (x + 4)^2 - 13$$

Why do we prefer to have the square completed?

- It is easier to graph the function.
  - Since we know the shape of x^2 we can now use linear transformations to graph the new function.
  - We can automatically read off the vertex of the function. (If it looks like $(x - b)^2 + c$ then the vertex is (b, c))
  - We can automatically know the line of symmetry. (If it looks like $(x - b)^2 + c$ then the line is $x = b$)
- We can find the roots easily by solving for x.

The Quadratic Equation

Another advantage of completing the square is that it leads us to the familiar formula telling us the roots of any quadratic function. How do we derive it from a general quadratic polynomial? We complete the square and then we solve for x!

Here is a general quadratic polynomial equation: $ax^2 + bx + c = 0$

First let's divide through by a so that our polynomial is monic and easier to work with. Now we have:

$$0 = x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2$$

So now, since the expression was equal to zero we have:

$$\left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

Taking the square root of both sides we have:

$$x+\frac{b}{2a}=\pm\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}$$

With common denomonators this looks like:

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Solving for x we have:

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

And simplifying the equation gives us the most familiar form:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

We need to recognize that at the step in which we take the square root of both sides, we need to add the $\pm$ because we have to account for both the positive and negative solution.

Fractions and Decimals

We know we can express rational numbers in decimal expansions, here are a few examples:

$\frac{1}{2}=.5$, $\frac{1}{3}=.333\overline{3}$, $d\frac{1}{4}=.25$, $\frac{1}{5}=.2$, $\frac{1}{6}=.166\overline{6}$, $\frac{1}{7}=.142857\overline{142857}$

We've been taught a relationship between rational numbers and their decimal expansions, that is to say:

$\mathbf{Q} = \{$ repeating or terminating decimals $\}$

How do we show that a repeating or terminating decimal is in fact a rational number? If the decimal is terminating that is an easy task. First let's do an example:

Let our decimal be .112 then we know we can write that as the fraction $\frac{112}{1000}$ which can be simplified to $\frac{14}{125}$.

Now let's take a general terminating decimal, $.a_1a_2...a_s$. How do we turn it into a fraction?

We let $.a_1a_2...a_s = \dfrac{k}{10^s}$ where $k = a_1a_2...a_s \in \mathbf{N}$ and $s \in \mathbf{N}$

What if the decimal is repeating? Let's do an example first with $a = .166\overline{6}$:

FIrst consider $\dfrac{a}{10} = .0166\overline{6}$

Then we subtract a and $\dfrac{a}{10}$ to get $a - \dfrac{a}{10} = .15$

So we have $\dfrac{9a}{10} = .15$ and we know from the above work that $.15 = \dfrac{15}{100}$

Then, solving for a we have $a = \dfrac{15}{90}$ which simplifies to $a = \dfrac{1}{6}$

But how would be do this in general? Let's make an algorithm for putting a repeating decimal into fractional form.

Repeating Decimal into Fraction Algorithm

1. Call the repeating decimal a and note the number of places there are before the repeating part (denote it m) and how long the repeating part is (call this the period, denote it k). Note that k, $m \in \mathbf{N}$
2. Divide a by 10^k. We do this so that repeating parts are "lined up".
3. Subtract a and $\frac{a}{10^k}$ so that we have $a\left(\frac{10^k-1}{10^k}\right)$ = some terminating decimal. In fact we can call this terminating decimal $\frac{b}{10^{k+m}}$ where $b \in \mathbf{Z}$.

4. Now solve for a. This gives us $a = \frac{10^m b}{10^k - 1} \in \mathbf{Q}$

So we have shown that $\{$ repeating or terminating decimals $\} \subseteq \mathbf{Q}$

Now we need to show that $\mathbf{Q} \subseteq \{$ repeating or terminating decimals $\}$. First let us deal with fractions that are equal to a terminating decimal.

We only need to deal with the case that our fraction looks like $\frac{1}{q}$ because we can just multiply a terminating decimal by an integer and still get a terminating decimal. Let's hypothosize that q must be made up of divisors of 10. This is because we know all terminating decimals can be represented as fractions with a power of 10 in the denominator.

So we have $q = 2^m 5^n$ where $m, n \in \mathbf{N}$. Let's show that we can write $\frac{1}{q}$ as a terminating decimal. Let $s = max\{m, n\}$. Then we have:

$$\frac{1}{q} = \frac{k}{10^s} \text{ where } k = \frac{10^s}{2^m 5^n} \in \mathbf{N}$$

But what if q were made up of more than the divisors of 10? We wouldn't be able to write this as a terminating decimal because we could never write the denominator as a power of 10.

How do we know all other forms of q will lead us to a repeating decimal? Well our choices for remainders when we do the division are $\{0, 1, 2, ..., q-1\}$. If we ever get 0 for a remainder than we have a terminating decimal and we know q is just powers of 2 and 5. If we never get 0 than we only have $\{1, 2, ..., q-1\}$ choices for remainders. Since there are only a finite number of these remainders, at some point one must repeat and the sequence of numbers that were remainders after the repeating one would start over again. So we would get a repeating decimal.

So we have that $\mathbf{Q} = \{$ repeating or terminating decimals $\}$.