MAE 301/501 Final Exam

There are 3 problems in this exam. It is a take-home exam, due on the scheduled date of the final (May 19, 2009) by 5pm.
Since this is a course for future teachers, the clarity of your answer to each of the questions is as important as its correctness. Please make sure that your solutions are well thought out, properly justified, and easy to follow. Being correct is good, too.
Obviously, you should do your own work and not get help from others.

1. In class, we discussed the long division of polynomials and its relationship to "synthetic division". The standard method only works with a linear divisor (of the form $x-c$); we discussed how to adapt the method to a higher degree divisor of the form $x^n - c$.

   Figure out a way to adapt the method to an arbitrary divisor (you may assume the divisor is monic, i.e. it is of the form $x^n + ax^{n-1} + \ldots + c$). Give a explanation of your method and why it works. Illustrate your discussion with some well-chosen examples.

2. Let f be a function from $\mathbb{Q}$ to $\mathbb{R}$ such that the following holds for all x and y in the domain:

   $$f(x+y) = f(x)f(y) \qquad\qquad f(1) = a$$

   (If you like, you may take $a = 2$, but don't use any special properties of 2. Just to be explicit, if $a = 2$, we have $f(4) = f(1) \cdot f(3) = 2f(3)$.) Thus, f is a homomorphism from the additive group of rationals to (a subgroup of) the multiplicative group of the reals (although you'd need to show $f(0) = 1$, which is true).

   (a) What (explicitly) is this function f? Write a formula for f, and **fully** justify your answer.

   (b) Suppose we want to extend the domain of f to $\mathbb{R}$ in such a way that it agrees with the answer to the previous part. What additional assumption(s) must you make about f to ensure that the answer you gave in the first part is the only possible answer? Can you give two different functions which satisfy the constraints of the problem, and agree on $\mathbb{Q}$ but differ on $\mathbb{R}$? Again, fully justify your answer.

   (c) (*optional, extra credit*) What is the largest subset of $\mathbb{R}$ for which the answer to the first part works without any additional assumptions?

3. Here is a problem adapted from a high-school text:

   > Out of all the lines passing through the point $(2,5)$, find the one which cuts off the triangle of smallest area in the first quadrant.

   (a) Solve this problem in such a way that your solution can be understood by a good high-school student who has not taken calculus. Of course, you can use calculus to check your answer, but your solution must not use calculus. You may use either an algebraic approach (using coordinates) or a geometric approach. Make sure your answer is well written, and that your method is easy to follow. (Bonus: combine both the algebraic and geometric approaches.)

   (b) Generalize your answer to an arbitrary point (p,q) instead of $(2,5)$. If you prefer to do both parts simultaneously, using $p = 2, q = 5$ as an example, that's fine. But make sure your answer is clear and well-written. I may ask my 17-year-old son to read your answer.