MAT-550 Schul　　　　Midterm　　　　March , 2016

First Name: ____________________

Last Name: ____________________

Stony Brook ID: ____________________

Signature: ____________________

Write coherent mathematical statements and show your work on all problems. If you use a theorem from the book, you must fully state it. If you give an example/construction then you must prove it is such. Please write clearly.

Rules.

1. Start when told to; stop when told to.
2. No notes, books, etc,...
3. Turn OFF all unauthorized electronic devices (for example, your cell phone).

1 (30pts)		2 (10pts)	3 (10pts)	4 (10pts)	5 (10pts)

DO quesion number 1 (FIVE of the SIX parts).
Choose 3 out of the 4 questions numbered 2-5.

1. (**Do FIVE of the SIX parts of QUESTION 1.** Each part is worth 6 points)

(A.) Suppose that $f \in C[0,1]$ is a real valued continuous function on $[0,1]$, such that

$$\int_0^1 x^n f(x)dx = 0$$

for all nonnegative integers n. Show that $f(x) = 0$ for all $x \in [0,1]$?

[Answer sketch: Apply Stone-Weierstrass (or Weierstrass) to get that $f = \lim p_n$ where p_n is a polynomial and the limit is uniform. One can argue that

$$\int f^2 = \lim \int p_n \cdot f = 0$$

]

(B.) State the Open Mapping Theorem

(C.) State the Closed Graph Theorem

(D.) Define Weak L^p

(E) Give an example of a function f which is Weak L^3, but $f \notin L^3$.

[Answer sketch: Take $f(x) = \frac{1}{x^{\frac{1}{3}}}$, defined on $[1,\infty]$ and Lebesgue measure. Then $f \notin L^3$ since $\int_1^\infty x^{-1}dx = \infty$. However

$$\mu\{x \geq 1 : f(x) > \alpha\} = \mu\{x \geq 1 : x < \alpha^{-3}\} < \alpha^{-3}.$$

]

(F) State one of the two interpolation theorems we learned in class

2. (10 points)
Suppose X is a Banach space. Show that if X^* is separable, then X is separable.

[Answer sketch: Let $\{f_n\}_1^\infty \subset X^*$ be dense in X^*. Let x_n be such that $\|x_n\| = 1$ and $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Let $V = \text{spn}\{x_1, x_2, ...\}$. If the closure of V, $\bar{V}$, satisfies $\bar{V} = X$ then we are done. Otherwise, let $v \in X \setminus \bar{V}$. By an application of Hahn-Banach (specifically, Thm 5.8) we have that there is $f \in X^*$ with $f(v) > 0$ and $f|_{\bar{V}} = 0$. In fact, we may assume $\|f\| = 1$ WLOG. Take f_n such that $|f - f_n| < \frac{1}{5}$. Then, $\|f_n\| > \frac{4}{5}$. We have

$$\frac{2}{5} \leq \frac{1}{2}\|f_n\| \leq |f_n(x_n)| = |f_n(x_n) - f(x_n)| \leq \frac{1}{5}$$

which is a contradiction.]

3. (10 points) (You may use the open mapping theorem below.)

(a) Prove the Closed Graph Theorem

(b) Let $\ell^p(\mathbb{N},\mathbb{R}) := \{(x_1, x_2, ...) : \ x_i \in \mathbb{R}, \ \sum x_i^p < \infty\}$ and for $x \in \ell^p(\mathbb{N},\mathbb{R})$, set $\|x\|_p = (\sum |x_i|^p)^{\frac{1}{p}}$. Note that for $x \in \ell^1(\mathbb{N},\mathbb{R})$ we have $\|x\|_7 \le \|x\|_1$. Is it true that there is a $C > 0$ such that $\|x\|_1 \le C\|x\|_7$ for all $x \in \ell^1(\mathbb{N},\mathbb{R})$?

[Answer sketch: (b) First note that $\ell^1(\mathbb{N},\mathbb{R}) \subset \ell^7(\mathbb{N},\mathbb{R})$. Now, One could try to argue using the identity map

$$i : (\ell^1(\mathbb{N},\mathbb{R}), \|\cdot\|_1) \to (\ell^1(\mathbb{N},\mathbb{R}), \|\cdot\|_7)$$

which is bounded, and thus **seems** open (by the Open Mapping Thm). BUT THIS IS WRONG, since $(\ell^1(\mathbb{N},\mathbb{R}), \|\cdot\|_7)$ is not complete, as can be seen by taking $a^n \in \ell^1(\mathbb{N},\mathbb{R})$ given by $a^n = (1, \frac{1}{2}, ..., \frac{1}{n}, 0, ..., 0, ...)$ and noting that a^n is Cauchy with $\|\cdot\|_7$, but has no limit (any limit would also be an $\ell^7(\mathbb{N},\mathbb{R})$ limit, which we know is unique and is given by $(1, \frac{1}{2}, ..., \frac{1}{n}, ...)$.

More directly, if one had the inequlaity $\|x\|_1 \le C\|x\|_7$, then any $\ell^7(\mathbb{N},\mathbb{R})$ sequence would be in $\ell^1(\mathbb{N},\mathbb{R})$, but this is false since $(1, \frac{1}{2}, ..., \frac{1}{n}, ...) \in \ell^7$ and not in $\ell^1(\mathbb{N},\mathbb{R})$.]

4. (10 points)
Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a Lipschitz vector field. Suppose also that the maximal integral curves of this (time-independent) vector field are defined on the entire real line, i.e. $\Phi_F^t : \mathbb{R}^2 \to \mathbb{R}^2$ is defined for all $t \in \mathbb{R}$. Show that there is a constant $K > 0$ such that

$$\|\Phi_F^t(x) - \Phi_F^t(y)\| \leq e^{Kt}\|x - y\| \, .$$

[Answer sketch: repeat proof of Lemma 6.3 in the notes with K being the Lipschitz constant for F.]

5. (10 points)
Let $\epsilon > 0$ be given. Let $f : [0,2] \to \mathbb{R}$ be continuous. Show that there is a polynomial p such that $p(k) = f(k)$ for $k \in \{0,1,2\}$ and for all $x \in [0,2]$ we have $|f(x) - p(x)| < \epsilon$.

[Answer sketch: Let

$$a_0(x) = (x-1)(x-2)/2$$
$$a_1(x) = -x(x-2)$$
$$a_2(x) = x(x-1)/2.$$

So for integers $k, n \in \{0,1,2\}$, we have $a_n(k) = 1$ iff $k = n$ and zero otherwise. Also note that $|a_n(x)| \leq 2$ for all $x \in [0,2]$.

Approximate f by a polynomial q such that $\|f - q\|_u < \epsilon/10$ (using Stone-Weierstrass). Let $\delta_k = f(k) - q(k)$, so that $|\delta_k| < \epsilon/10$. Then set

$$p(x) = q(x) + \delta_0 a_0(x) + \delta_1 a_1(x) + \delta_2 a_2(x).$$

We have that p is a polynomial, and p agrees with f at $\{0,1,2\}$. We also have that

$$\|f - p\|_u \leq \|f - q\|_u + (\sup_k \|a_k\|_u) \sum_k |\delta_k| \leq (1 + 2 \times 3)\epsilon/10 < \epsilon$$

Note: a similar argument can be used for any fixed number of points in a closed interval...

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