MAT-544 Schul — Final — December , 2015

First Name: ____________________

Last Name: ____________________

Stony Brook ID: ____________________

Signature: ____________________

Write coherent mathematical statements and show your work on all problems. If you use a theorem from the book, you must fully state it. If you give an example/construction then you must prove it is such. Please write clearly.

Rules.

1. Start when told to; stop when told to.
2. No notes, books, etc,...
3. Turn OFF all unauthorized electronic devices (for example, your cell phone).

1 (10pts)	2 (20pts)	3 (10pts)	4 (10pts)

5 (10pts)	6 (10pts)	7 (20pts)	8 (20pts)

Choose 6 out of the 8 questions. Note that they are not all worth the same amount of points. Some questions are hard.

1. (10 points)

(a) What is a simple function?

(b) Let $f : \mathbb{X} \to [0, \infty)$ be a measurable function. Show by construction that there is a sequence of *simple* functions $\phi_n : \mathbb{X} \to \mathbb{R}$ such that $\phi_n \to f$ everywhere, and that this limit is uniform on the set $\{x : f(x) < 1000\}$.

2. (20 points) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of Lebesgue-integrable functions. For each of the statements below, prove it or find a counterexample.

(a) (4 points) If $f_n \to f$ almost everywhere, then a subsequence converges to f in L^3.

(b) (4 points) If $f_n \to f$ in measure, then $f_n \to f$ in L^3.

(c) (4 points) If $f_n \to f$ in L^3, then a subsequence converges to f in measure.

(d) (8 points) If $f_n \to f$ in measure, then a subsequence converges to f almost everywhere.

3. (10 points) For an integer j, let

$$\Delta_j = \{[\frac{i}{2^j}, \frac{i+1}{2^j}) : i \in \mathbb{Z}\}\,.$$

Let

$$\Delta = \bigcup_{j=0}^{\infty} \Delta_j\,.$$

Fix a finite Borel measure μ on $\mathbb{R}$, and for an integrable function $f : [0, 1) \to \mathbb{R}$, let

$$M_\Delta f(x) = \sup_{\substack{I \in \Delta \\ I \ni x}} \frac{1}{\mu(I)} \int_I |f| d\mu\,.$$

Show that for any $\lambda > 0$,

$$\mu\{x \in [0, 1) : M_\Delta f(x) > \lambda\} < \frac{1}{\lambda} \int_{[0,1)} |f| d\mu\,.$$

4. (10 points) Use the same definitions as the previous question. For $x \in [0, 1)$, and $n \in \mathbb{N}$, let $I_n(x)$ be the unique interval in Δ_n such that $I_n(x) \ni x$. Let $f : \mathbb{R} \to \mathbb{R}$ be a locally integrable function. Let

$$A_n f(x) = \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f d\mu\,.$$

Use the result of the previous question to show that for μ almost every $x \in [0, 1)$ we have

$$\lim_{n\to\infty} A_n f(x) = f(x)\,.$$

Note: You may use without proof the following fact. For any $\epsilon > 0$ there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $\|g - f\|_1 < \epsilon$ and g is a finite linear combinations of characteristic functions of intervals belonging to Δ, i.e. g can be written in the form $\sum_{j=1}^{k} a_i \chi_{I_j}$ where $I_j \in \Delta$.

5. (10 points) Let $f \in L^1(\mathbb{X}, \mathcal{M}, \mu) \cap L^\infty(\mathbb{X}, \mathcal{M}, \mu)$.

   (a) Show that for any $q \in (1, \infty)$ we have $f \in L^q$.

   (b) Show that $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

6. (10 points) Suppose $1 \le p < \infty$. Show that if $T : L^p(\mathbb{X}, \mathcal{M}, \mu) \to \mathbb{R}$ is a bounded linear operator, then there is a measure ν on $(\mathbb{X}, \mathcal{M})$ such that $\nu \ll \mu$ and $T(f) = \int f d\nu$ for all $f \in L^p$. You may assume that $\mu(\mathbb{X}) < \infty$.

7. (20 points)
   (a) (5 points) State Egoroff's theorem.
   (b) (5 points) State Lusin's theorem.
   (c) (10 points) Use Egoroff's theorem to derive Lusin's theorem. Note: Upon request, I can give you the answer to (a)+(b) and deduct 10 points from your total.

8. (20 points) Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a function such that for any rational $q \in \mathbb{Q}$ and any $x \in \mathbb{R}$ we have $F(x + q) = F(x)$.
   Show that if F is measurable then there exists $c \in \mathbb{R}$ such that $F(x) = c$ almost everywhere with respect to Lebesgue measure.