Below is a solution to ONE OF THE VERSIONS of the exam. There were four versions in all. The differences were minor.

1. (40 points)
Calculate the following integrals. Even if you can do this in your head, write some stuff along the way. This is your way to "show your work".

(i)
$$\int_2^4 x^3 + \frac{3}{x^2} + e^x dx$$

[Answer:

$$\begin{aligned}\int_2^4 x^3 + \frac{3}{x^2} + e^x \, dx &= \int_2^4 x^3 + 3x^{-2} + e^x \, dx \\ &= \left[\frac{1}{4}x^4 - 3x^{-1} + e^x\right]_2^4 \\ &= \left(\frac{1}{4} \cdot 4^4 - 3 \cdot 4^{-1} + e^4\right) - \left(\frac{1}{4} \cdot 2^4 - 3 \cdot 2^{-1} + e^2\right) \\ &= \boxed{\frac{243}{4} + e^4 - e^2}\end{aligned}$$

Common mistakes: misuse of the power rule on the x^{-2} term.]

(ii)
$$\int_{-1}^1 \frac{2}{1+t^2} dt$$

[Answer:

$$\begin{aligned}\int_{-1}^1 \frac{2}{1+t^2} dt &= 2\int_{-1}^1 \frac{1}{1+t^2} dt \\ &= 2\left[\arctan(t)\right]_{-1}^1 \\ &= 2\left(\arctan(1) - \arctan(-1)\right) \\ &= 2\left(\frac{\pi}{4} + \frac{\pi}{4}\right) \\ &= \boxed{\pi}\end{aligned}$$

Common mistakes: not recognizing $1/(1+t^2)$ as the derivative of $\arctan(t)$, and mistaking $\arctan(t) = 1/\tan(t)$. Remember that arctan is the *functional* inverse of tan, i.e. $\tan(\arctan(t)) = t$.]

(iii)

$$\int_0^3 |1-2t| dt.$$

[Answer: The expression $1-2t$ is positive for $t < 1/2$ and negative for $t > 1/2$.

$$\begin{aligned}
\int_0^3 |1-2t|\,dt &= \int_0^{1/2} (1-2t)\,dt + \int_{1/2}^3 -(1-2t)\,dt \\
&= \left[t - t^2\right]_0^{1/2} - \left[t-t^2\right]_{1/2}^3 \\
&= \left(\frac{1}{2} - \left(\frac{1}{2}\right)^2\right) - \left[(3-3^2) - \left(\frac{1}{2} - \left(\frac{1}{2}\right)^2\right)\right] \\
&= \boxed{\frac{13}{2}}
\end{aligned}$$

Common mistakes: not splitting the integral, or splitting the integral incorrectly.

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(iv)

$$\int \frac{\sin(t)}{\cos^3(t)} dt.$$

[Answer: Substitute $u = \cos(t)$, so that $du = -\sin(t)\,dt$. Then

$$\begin{aligned}
\int \frac{\sin(t)}{\cos^3(t)}\,dt &= \int \frac{-1}{u^3}\,du \\
&= \frac{1}{2u^2} + C \\
&= \frac{1}{2\cos^2(t)} + C \\
&= \boxed{\frac{\sec^2(t)}{2} + C}
\end{aligned}$$

If one reinterprets the integrand as $\tan(t)\sec^2(t)$, then the natural u substitution is $u = \tan(t)$. In this case the answer is $\frac{\tan^2(t)}{2} + C$, which is a seemingly different answer. However, the identity $\sin^2(t) + \cos^2(t) = 1$ shows that $\tan^2(t) + \sec^2(t) = 1$, i.e. $\tan^2(t) = \sec^2(t) - 1$. Since the indefinite integral is only defined up to a constant, both answers are in agreement.

Common mistakes: choosing the u substitution incorrectly, or failing to carry through the substitution completely (i.e. the integrand should never contain both t and u).]

2. (30 points)
A runner is running along a long and perfectly straight road. At time $t = 0$ the runner starts measuring their speed. Below is a table of the speed at various times.

Time in seconds	$t = 0$	$t = 2$	$t = 4$	$t = 6$	$t = 8$	$t = 10$
Speed in meters/sec	6	7	5	8	4	7

Give both an overestimate and an underestimate for the distance the runner covered. Explain your answer carefully.

[Answer: We will assume that the runner's speed does not fluctuate beyond the max and min of the endpoint values on an interval. For instance, in the first interval ($t = 0$ to 2) the runner is assumed have speed between 6 m/s and 7 m/s. Therefore taking 6 m/s throughout the interval yields an underestimation, and taking 7 m/s yields an overestimation. In general, the lesser of the two endpoints gives the underestimate and the greater gives the overestimate.

The underestimates for the 5 intervals are then 6, 5, 5, 4, 4 m/s. Then the total distance covered is

$$(6 \text{ m/s})(2 \text{ s})+(5 \text{ m/s})(2 \text{ s})+(5 \text{ m/s})(2 \text{ s})+(4 \text{ m/s})(2 \text{ s})+(4 \text{ m/s})(2 \text{ s}) = 48\text{m}.$$

The overestimates are 7, 7, 8, 8, 7 m/s, so we get

$$(7 \text{ m/s})(2 \text{ s})+(7 \text{ m/s})(2 \text{ s})+(8 \text{ m/s})(2 \text{ s})+(8 \text{ m/s})(2 \text{ s})+(7 \text{ m/s})(2 \text{ s}) = 74\text{m}.$$

Common mistakes: systematically using only the left or right endpoints. This only works when the data in question is monotonic, i.e. always increasing or always decreasing.

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3. (30 points)
Consider the limit of the sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\left(\frac{5i}{n}\right)^3.$$

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(i) Find a function $f(x)$ and an interval $[a, b]$ such that the limit above is the area under the graph of $y = f(x)$ over the interval $[a, b]$.

(ii) Use the following formula to calculate the above limit. (Do not compute the integral directly.)

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

[Answer: (i) We interpret the sum as a Riemann sum. Let $5/n$ be the width of each rectangle and $(5i/n)^3$ be the height. This corresponds to an interval of total width 5, and as i ranges from 1 to n, the expression $5i/n$ ranges approximately from 0 to 5. Therefore we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\left(\frac{5i}{n}\right)^3 = \int_0^5 x^3\,dx.$$

(ii) Since 5 and n are independent of the indexing variable i, we can factor them out of the summation and then apply the given formula.

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\left(\frac{5i}{n}\right)^3 &= \lim_{n\to\infty} \frac{5^4}{n^4}\sum_{i=1}^{n} i^3 \\ &= \lim_{n\to\infty} \frac{5^4}{n^4}\left(\frac{n(n+1)}{2}\right)^2 \\ &= \lim_{n\to\infty} \frac{5^4}{2^2}\left(\frac{n^2(n+1)^2}{n^4}\right)\end{aligned}$$

This expression is a ratio of two polynomials of the same degree, so the limit as $n \to \infty$ is the ratio of their leading coefficients. The numerator has leading coefficient 5^4 and the denominator has leading coefficient 4. Therefore the final answer is $\boxed{5^4/4.}$

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4. (30 points)

(i) Find the derivative $g'(x)$ of the function $g(x) = \int_0^{\sin(x)} t^2 + t dt$.

(ii) Find the derivative $h'(x)$ of the function $h(x) = \int_x^5 e^{\sin(t^3)} dt$.

(iii) Use the FTC (Fundamental theorem of calculus) to define a function $f(x)$ such that
$f'(x) = \sin(x)^{34} x^5$ and $f(4) = 0$.

Note: you can leave your answer in the form of an integral.

[Answer:

(i) By the FTC, $g(x) = G(\sin(x)) - G(0)$ where $G(u)$ is an antiderivative of $u^2 + u$. In other words, $G'(u) = u^2 + u$. Then

$$\begin{aligned} g'(x) &= [G(\sin(x)) - G(0)]' \\ &= G'(\sin(x))\cos(x) \\ &= \boxed{(\sin^2(x) + \sin(x))\cos(x).} \end{aligned}$$

Here we have used the chain rule on $G(\sin(x))$, and the fact that $G(0)$ is a constant and so has zero derivative.

Common mistakes: difficulty applying the chain rule.

(ii) You can write

$$h(x) = \int_x^5 e^{\sin(t^3)} dt = \int_0^5 e^{\sin(t^3)} dt - \int_0^x e^{\sin(t^3)} dt.$$

And then proceed as above. Another way is to note that

$$h(x) = -\int_5^x e^{\sin(t^3)} dt.$$

Either way, by the FTC we immediately get

$$h'(x) = -e^{\sin(x^3)}.$$

(iii) The FTC says that a function of the form $f(x) = \int_a^x \sin(t)^{34} t^5 \, dt$, for any constant a, will have derivative $f'(x) = \sin(x)^{34} x^5$. The condition $f(4) = 0$ can be obtained by setting $a = 4$. Therefore

$$\boxed{f(x) = \int_4^x \sin(t)^{34} t^5 \, dt.}$$

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5. (30 points). Calculate

(i)

$$\int \frac{1}{x(x-5)} dx$$

(ii)

$$\int x^2 e^{3x} dx$$

[Answer:

(i) The first step is to reexpress the integrand using a partial fractions decomposition. Setting

$$\frac{1}{x(x-5)} = \frac{A}{x} + \frac{B}{x-5},$$

multiplying through by $x(x-5)$ yields the equation

$$1 = A(x-5) + Bx.$$

Collect powers of x to get $1 = (A+B)x - 5A$. Interpreting both sides of the equation as a polynomial in x, we can match coefficients to get the two equations

$$A + B = 0, \quad -5A = 1.$$

The solutions are $A = -1/5, B = 1/5$. Now we can evaluate the integral.

$$\int \frac{1}{x(x-5)} \, dx = \int \frac{-1/5}{x} + \frac{1/5}{x-5} \, dx$$
$$= \boxed{-\frac{1}{5} \ln|x| + \frac{1}{5} \ln|x-5| + C}$$

(ii) This integral is calculated by repeated application of integration by parts. First set $u = x^2, dv = e^{3x} \, dx$, so $du = 2x$ and $v = e^{3x}/3$. We get

$$\int x^2 e^{3x} \, dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x e^{3x} \, dx.$$

Now to calculate this second integral, set $u = x$ and $dv = e^{3x}\,dx$, so that $du = dx$ and $v = e^{3x}/3$. Then

$$\begin{aligned}\int xe^{3x}\,dx &= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C.\end{aligned}$$

All together,

$$\begin{aligned}\int x^2e^{3x}\,dx &= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C\right) \\ &= \boxed{\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.}\end{aligned}$$

Common mistakes: making the wrong choice for u and dv. In general, u should be the term that becomes simpler when differentiated. For instance, polynomials like x^2 above become simpler because the degree decreases.

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