1. (10 points) Find the limit if it exists, or show that it does not:

(i)

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+3y^4}$$

(ii)

$$\lim_{(x,y)\to(0,0)} \frac{x^6}{x^4+3y^4}$$

i) Along $x=0$, $\lim\limits_{y\to 0} \frac{(0)^4}{(0)^4+3y^4} = \lim\limits_{y\to 0} 0 = 0$,

along $y=0$, $\lim\limits_{x\to 0} \frac{x^4}{x^4+3(0)^4} = \lim\limits_{x\to 0} 1 = 1$,

Since we get different limits along two different paths,
$\lim\limits_{(x,y)\to(0,0)} \frac{x^4}{x^4+3y^4}$ does not exist.

ii) For all $(x,y) \neq (0,0)$, $0 \leq \frac{x^4}{x^4+3y^4} \leq 1$

$$\Rightarrow 0 \leq \frac{x^6}{x^4+3y^4} = x^2 \cdot \frac{x^4}{x^4+3y^4} \leq x^2$$

Since $\lim\limits_{(x,y)\to(0,0)} 0 = 0 = \lim\limits_{(x,y)\to(0,0)} x^2$,

by Squeeze Theorem $\lim\limits_{(x,y)\to(0,0)} \frac{x^6}{x^4+3y^4} = 0$

2. (10 points) Let

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(i) Find $f_x(0,0)$.

(ii) Is f_x a continuous function?

i) $$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h\cdot(0)^2}{h^2+(0)^2} - 0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

ii) Note that $$f_x(x,y) = \begin{cases} \frac{y^2(x^2+y^2) - xy^2(2x)}{(x^2+y^2)^2} = \frac{y^4 - x^2y^2}{(x^2+y^2)^2} & \text{for } (x,y) \neq (0 \\ 0 & \text{for } (x,y) = (0 \end{cases}$$

Along $x=0$, $\lim_{y\to 0} \frac{y^4 - (0)^2y^2}{((0)^2+y^2)^2} = 1$,

along $y=0$, $\lim_{x\to 0} \frac{(0)^4 - x^2(0)^2}{(x^2+(0)^2)^2} = 0$.

Hence $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist

$\Rightarrow$ f_x is not continuous at $(0,0)$

$\Rightarrow$ f_x is not continuous

3. (10 points) Let

$$g(x,y) = 2x^3y + 2xy^5\,.$$

Find:

(i) $\frac{\partial^2 g}{\partial x \partial y}$

(ii) g_{xx}

i) $$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(2x^3 + 10xy^4)$$

$$= 6x^2 + 10y^4$$

ii) $$g_x = 6x^2y + 2y^5$$

$$g_{xx} = 12xy$$

4. (10 points) Find the tangent plane of $z = f(x, y)$ at the point $P = (2, 1, e^{4e})$, where

$$f(x, y) = e^{2xe^y}.$$

Equation : $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$P = (x_0, y_0, z_0) = (2, 1, e^{4e})$.

$$f_x(x_0, y_0) = 2e^{2xe^y} \cdot e^y \Big|_{(2,1)} = 2e^{4e+1}$$

$$f_y(x_0, y_0) = 2e^{2xe^y} \cdot xe^y \Big|_{(2,1)} = 4e^{4e+1}$$

$$z - e^{4e} = 2e^{4e+1}(x - 2) + 4e^{4e+1}(y - 1)$$

Common mistakes:

- Computed f_x or f_y wrong.
- Didn't plug (x_0, y_0) into f_x or f_y

5. (10 points)

A model for the surface area of a human body is given by

$$S = w^{\frac{1}{2}}h\,,$$

where w is the weight measured in pounds, h is the hight measured in feet, and S is the area in measured square feet. Assume that you have measured your nephew to have a weight of 100 pounds and a hight of 4 feet. Suppose you know that your scale has an error of up to 1 pound, and your tape measure has an error of up to $\frac{1}{10}$ ft. Use differentials to estimate the maximum error in the surface area of your nephew as given by the formula above. Why is the function in question differentiable?

$$\frac{\partial s}{\partial w} = \frac{1}{2}\cdot\left(w^{\frac{1}{2}}\right)^{-1}\cdot h = \frac{h}{2\sqrt{w}} \Longrightarrow \frac{\partial s}{\partial w}(100,4) = \frac{2}{10}$$

$$\frac{\partial s}{\partial h} = \sqrt{w} \Longrightarrow \frac{\partial s}{\partial h}(100,4) = 10$$

for $w > 0$, $\frac{\partial s}{\partial w}$ and $\frac{\partial s}{\partial h}$ are continuous functions

$\Longrightarrow$ S is differentiable. (Theorem 8 in Section 15.4)

Thus $\Delta S \approx dS = \frac{\partial s}{\partial w}\cdot \Delta w + \frac{\partial s}{\partial h}\cdot \Delta h = \frac{2}{10}\Delta w + 10\Delta h$

$\Longrightarrow |dS| \leq \frac{2}{10}\cdot 1 + 10\cdot\frac{1}{10} = 1.2$

$\Longrightarrow |\Delta S|$ is approxmatly bounded by 1.2 feet^2.

Common mistakes

(1) $S(w,h)$ continuous $\not\Longrightarrow$ $S(w,h)$ differentiable

(2) saying S_w and S_h are differentiable without justification

(3) saying S is a polynomial (it is not!)