The geometry of Radon-Nikodym Lipschitz differentiability spaces, II

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RNP Lipschitz differentiability spaces

Let \((X, d)\) be a metric space, \(V\) be a Banach space, and \(\varphi : X \to \mathbb{R}^n\) (a chart). We say \(f : X \to V\) is \(\varphi\)-differentiable at \(x_0 \in X\) if there exists a unique \(Df(x_0) \in L(\mathbb{R}^n, V)\) so that

\[
f(x) = f(x_0) + Df(x_0)(\varphi(x) - \varphi(x_0)) + o(d(x, x_0)).
\]

**Definition**

A metric measure space \((X, d, \mu)\) is a Radon-Nikodym Lipschitz differentiability space (RNP-LDS) if there is a Lipschitz chart \(\varphi : X \to \mathbb{R}^n\) so that for every Banach space \(V\) with the Radon-Nikodym property and every Lipschitz function \(f : X \to V\) is \(\varphi\)-differentiable at \(\mu\)-a.e. \(x \in X\).
Upper gradients

Let \((X, d, \mu)\) be a (rectifiably) path connected space. A measurable function \(\rho : X \to [0, \infty]\) is an upper gradient for a Lipschitz function \(f : X \to \mathbb{R}\) if for every rectifiable curve \(\gamma : [a, b] \to X\) we have

\[
|f(\gamma(b)) - f(\gamma(a))| \leq \int_\gamma \rho \, ds.
\]

Upper gradients are not unique, but they always exist for path connected spaces by considering \(\rho \equiv \infty\). If \(X\) is geodesic, then we may take \(\rho \equiv L\) when \(f\) is \(L\)-Lipschitz.
Poincaré inequality

**Definition (Heinonen-Koskela)**

A path connected space \((X, d, \mu)\) satisfies a Poincaré inequality if there exists \(C \geq 1\) and \(p \in [1, \infty)\) so that for every Lipschitz function \(f : X \to \mathbb{R}\) with upper gradient \(\rho : X \to [0, \infty]\), we have

\[
\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C r \left( \int_{B(x,Cr)} \rho^p \, d\mu \right)^{1/p}, \quad \forall x \in X, \, r > 0.
\]

“Path fluctuations control metric fluctuations”

A PI space is metric measure space that satisfies a Poincaré inequality and is doubling, *i.e.* there exists \(C \geq 1\) so that

\[
\mu(B(x,2r)) \leq C \mu(B(x,r)), \quad \forall x \in X, \, r > 0.
\]
PI and differentiability

**Theorem (Cheeger-Kleiner)**

PI spaces are Radon-Nikodym Lipschitz differentiability spaces.

Q: Does the converse hold? A: No.

PI spaces are path connected. Positive measure subsets of RNP-LDS are RNP-LDS (with induced measure and metric). Thus, fat Cantor sets of 

\[[0,1]\]

are RNP-LDS, but totally disconnected.

Need to relax doubling and PI.

Need to relax line integral and upper gradient.
Line integrals

Let \((X, d, \mu)\) be a metric measure space (possibly disconnected) and \(f : X \to \mathbb{R}\) be 1-Lipschitz and \(\rho : X \to [0, \infty]\) be measurable.

Let \(\gamma : K \to X\) be a 1-Lipschitz map so that \(K \subset \mathbb{R}\) is compact (a "fragment") with \(a = \min K\) and \(b = \max K\). We want something like

\[
|f(\gamma(b)) - f(\gamma(a))| \leq \int_\gamma \rho \ ds.
\]

We can still make sense of

\[
\int_\gamma \rho \ ds := \int_K \rho(\gamma(s))|\gamma'(s)| \ ds.
\]

However, \(f\) can fluctuate across gaps of \(X\) and thus \(K\).

Example: Let \(X \subset [0, 1]\) be a fat Cantor set and \(f(t) = \int_0^t \chi_{[0,1]\setminus X} \ dx\).
Line integrals (cont.)

Let \((c, d)\) be a gap in \(K\). As \(f\) and \(\gamma\) are 1-Lipschitz, we have

\[
|f(\gamma(c)) - f(\gamma(d))| \leq |c - d|.
\]

Define the \(*\)-integral

\[
\int_{\gamma}^* \rho := \int_{K} \rho \ ds + |[a, b]\setminus K|.
\]

We then have that

\[
|f(\gamma(b)) - f(\gamma(a))| \leq \int_{\gamma}^* 1.
\]

We say \(\rho : X \to [0, 1]\) is a \(*\)-upper gradient of \(f\) if for all fragments \(\gamma\)

\[
|f(\gamma(b)) - f(\gamma(a))| \leq \int_{\gamma}^* \rho.
\]
Asymptotic nonhomogeneous Poincaré inequality

A metric measure space \((X, d, \mu)\) satisfies an asymptotic nonhomogeneous Poincaré inequality if there exist \(C \geq 1\) and continuous increasing moduli \(\zeta_x, o_x : [0, \infty) \to [0, \infty)\) for \(\mu\)-a.e. \(x \in X\) so that for every 1-Lipschitz \(f : X \to \mathbb{R}\) with \(*\)-u.g. \(\rho : X \to [0, 1]\), we have

\[
\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq r \zeta_x \left( \int_{B(x,Cr)} \rho \, d\mu \right) + o_x(r).
\]

Here, \(\zeta_x\) and \(o_x\) satisfy

\[
\lim_{t \to 0} \frac{o_x(t)}{t} = 0, \quad \lim_{t \to 0} \zeta_x(t) = 0.
\]
Characterization of RNP-LDS

Definition
A metric measure space \((X, d, \mu)\) is pointwise doubling if for \(\mu\)-a.e. \(x \in X\),

\[
\limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty.
\]

Theorem (Bate-L.)
A metric measure space is a Radon-Nikodym Lipschitz differentiability space if and only if it is pointwise doubling and satisfies an asymptotic nonhomogeneous Poincaré inequality.