Removability, Rigidity of Circle Domains and Koebe’s Conjecture

Malik Younsi, Stony Brook University

2016 AMS Spring Meeting
Circle domains
Circle domains

Definition
Circle domains

**Definition**

A domain $\Omega$ in the Riemann sphere $\hat{\mathbb{C}}$ is a **circle domain** if every connected component of its boundary is either a round circle or a point.
Circle domains

Definition

A domain $\Omega$ in the Riemann sphere $\hat{\mathbb{C}}$ is a \textbf{circle domain} if every connected component of its boundary is either a round circle or a point.
A domain $\Omega$ in the Riemann sphere $\hat{\mathbb{C}}$ is a **circle domain** if every connected component of its boundary is either a round circle or a point.

The boundary of any circle domain contains at most countably many circles.
The Koebe Uniformization Conjecture
Conjecture (Koebe, 1909)

*Every domain $D \subset \hat{\mathbb{C}}$ is conformally equivalent to a circle domain.*
Conjecture (Koebe, 1909)

Every domain $D \subset \mathbb{C}$ is conformally equivalent to a circle domain.

The conjecture is true if $D$
The Koebe Uniformization Conjecture

**Conjecture (Koebe, 1909)**

*Every domain* $D \subset \hat{\mathbb{C}}$ *is conformally equivalent to a circle domain.*

The conjecture is true if $D$

- has finitely many boundary components (Koebe, 1918)
The Koebe Uniformization Conjecture

Conjecture (Koebe, 1909)

Every domain \( D \subset \hat{\mathbb{C}} \) is conformally equivalent to a circle domain.

The conjecture is true if \( D \)

- has finitely many boundary components (Koebe, 1918)
- has countably many boundary components (He–Schramm, 1993).
Uniqueness of the Koebe map
Uniqueness of the Koebe map

\[ f \]
Conformal rigidity
Conformal rigidity

**Definition**

A circle domain $\Omega \subset \hat{\mathbb{C}}$ is **conformally rigid** if every conformal map of $\Omega$ onto another circle domain is the restriction of a Möbius transformation.
Definition

A circle domain $\Omega \subset \hat{\mathbb{C}}$ is **conformally rigid** if every conformal map of $\Omega$ onto another circle domain is the restriction of a Möbius transformation.

$\Omega$ is conformally rigid if
Conformal rigidity

**Definition**

A circle domain $\Omega \subset \hat{\mathbb{C}}$ is **conformally rigid** if every conformal map of $\Omega$ onto another circle domain is the restriction of a Möbius transformation.

$\Omega$ is conformally rigid if

- it has finitely many boundary components (Koebe, 1918)
Conformal rigidity

**Definition**

A circle domain $\Omega \subset \hat{\mathbb{C}}$ is **conformally rigid** if every conformal map of $\Omega$ onto another circle domain is the restriction of a Möbius transformation.

$\Omega$ is conformally rigid if

- it has finitely many boundary components (Koebe, 1918)
- it has countably many boundary components (He–Schramm, 1993)
Conformal rigidity

**Definition**

A *circle domain* $\Omega \subset \hat{\mathbb{C}}$ is **conformally rigid** if every conformal map of $\Omega$ onto another circle domain is the restriction of a Möbius transformation.

$\Omega$ is conformally rigid if

- it has finitely many boundary components (Koebe, 1918)
- it has countably many boundary components (He–Schramm, 1993)
- it has $\sigma$-finite length boundary (He–Schramm, 1994).
Conformally removable sets
Conformally removable sets

**Definition**

Let $E \subset \mathbb{C}$ be compact.
Definition

Let $E \subset \mathbb{C}$ be compact. We say that $E$ is **conformally removable** if every homeomorphism of $\hat{\mathbb{C}}$ which is conformal outside $E$ is actually conformal everywhere (hence is a Möbius transformation).
**Definition**

Let $E \subset \mathbb{C}$ be compact. We say that $E$ is **conformally removable** if every homeomorphism of $\hat{\mathbb{C}}$ which is conformal outside $E$ is actually conformal everywhere (hence is a Möbius transformation).

- Removable: sets of $\sigma$-finite length, quasicircles, Hölder curves
Conformally removable sets

**Definition**

Let $E \subset \mathbb{C}$ be compact.

We say that $E$ is **conformally removable** if every homeomorphism of $\hat{\mathbb{C}}$ which is conformal outside $E$ is actually conformal everywhere (hence is a Möbius transformation).

- Removable: sets of $\sigma$-finite length, quasicircles, Hölder curves
- Non-removable: sets of positive area
Conformally removable sets

Definition

Let $E \subset \mathbb{C}$ be compact.

We say that $E$ is conformally removable if every homeomorphism of $\hat{\mathbb{C}}$ which is conformal outside $E$ is actually conformal everywhere (hence is a Möbius transformation).

- Removable: sets of $\sigma$-finite length, quasicircles, Hölder curves
- Non-removable: sets of positive area
- There exist removable sets of Hausdorff dimension two and non-removable sets of Hausdorff dimension one
Conformally removable sets

Definition

Let $E \subset \mathbb{C}$ be compact. We say that $E$ is **conformally removable** if every homeomorphism of $\hat{\mathbb{C}}$ which is conformal outside $E$ is actually conformal everywhere (hence is a Möbius transformation).

- Removable: sets of $\sigma$-finite length, quasicircles, Hölder curves
- Non-removable: sets of positive area
- There exist removable sets of Hausdorff dimension two and non-removable sets of Hausdorff dimension one
- The complement of any non-removable Cantor set is a non-rigid circle domain.
Rigidity conjecture
Conjecture (He–Schramm 1994)
Conjecture (He–Schramm 1994)

\[ \text{Let } \Omega \text{ be a circle domain.} \]
Conjecture (He–Schramm 1994)

Let \( \Omega \) be a circle domain. The following are equivalent:
Conjecture (He–Schramm 1994)

Let $\Omega$ be a circle domain. The following are equivalent:

(A) $\Omega$ is conformally rigid;

---

Malik Younsi, Stony Brook University
Removability, Rigidity of Circle Domains and Koebe's Conjecture
Conjecture (He–Schramm 1994)

Let $Ω$ be a circle domain. The following are equivalent:

(A) $Ω$ is conformally rigid;

(B) The boundary of $Ω$ is conformally removable;
Conjecture (He–Schramm 1994)

Let $\Omega$ be a circle domain. The following are equivalent:

(A) $\Omega$ is conformally rigid;

(B) The boundary of $\Omega$ is conformally removable;

(C) Every Cantor set contained in the boundary of $\Omega$ is conformally removable.
Conjecture (He–Schramm 1994)

Let $\Omega$ be a circle domain. The following are equivalent:

(A) $\Omega$ is conformally rigid;
(B) The boundary of $\Omega$ is conformally removable;
(C) Every Cantor set contained in the boundary of $\Omega$ is conformally removable.

Remarks:
Conjecture (He–Schramm 1994)

Let $\Omega$ be a circle domain. The following are equivalent:

(A) $\Omega$ is conformally rigid;
(B) The boundary of $\Omega$ is conformally removable;
(C) Every Cantor set contained in the boundary of $\Omega$ is conformally removable.

Remarks:

- $(B) \Rightarrow (C)$ is trivial
Conjecture (He–Schramm 1994)

Let $\Omega$ be a circle domain. The following are equivalent:

(A) $\Omega$ is conformally rigid;
(B) The boundary of $\Omega$ is conformally removable;
(C) Every Cantor set contained in the boundary of $\Omega$ is conformally removable.

Remarks:

- (B) $\Rightarrow$ (C) is trivial
- If there are no circles in $\partial \Omega$, then (A) $\Rightarrow$ (B).
First Main Theorem
First Main Theorem

Theorem (Y. (2015))
Theorem (Y. (2015))

Let $\Omega$ be a circle domain whose boundary is the union of countably many circles and countably many totally disconnected compact sets.
Theorem (Y. (2015))

Let $\Omega$ be a circle domain whose boundary is the union of countably many circles and countably many totally disconnected compact sets.

Then (B) and (C) are equivalent.
Theorem (Y. (2015))

Let \( \Omega \) be a circle domain whose boundary is the union of countably many circles and countably many totally disconnected compact sets.

Then (B) and (C) are equivalent.

- The assumption holds if boundary circles don’t accumulate too much on point boundary components.
Theorem (Y. (2015))

Let $\Omega$ be a circle domain whose boundary is the union of countably many circles and countably many totally disconnected compact sets.
Then (B) and (C) are equivalent.

- The assumption holds if boundary circles don’t accumulate too much on point boundary components.
- The assumption does not hold if boundary circles accumulate everywhere.
A Sierpinski-type circle domain
Countable unions of certain conformally removable sets
Countable unions of certain conformally removable sets

The theorem is a consequence of the following more general result.
The theorem is a consequence of the following more general result.

Theorem (Y. (2015))
The theorem is a consequence of the following more general result.

**Theorem (Y. (2015))**

Let $E$ be a compact plane set of the form

$$E = \bigcup_{j=1}^{\infty} \Gamma_j \cup \bigcup_{k=1}^{\infty} E_k,$$

where each $\Gamma_j$ is a quasicircle and each $E_k$ is a totally disconnected compact set.
The theorem is a consequence of the following more general result.

**Theorem (Y. (2015))**

Let $E$ be a compact plane set of the form

$$E = \bigcup_{j=1}^{\infty} \Gamma_j \cup \bigcup_{k=1}^{\infty} E_k,$$

where each $\Gamma_j$ is a quasicircle and each $E_k$ is a totally disconnected compact set.

Then $E$ is conformally removable if and only if every $E_k$ is conformally removable.
The theorem is a consequence of the following more general result.

**Theorem (Y. (2015))**

Let $E$ be a compact plane set of the form

$$E = \bigcup_{j=1}^{\infty} \Gamma_j \cup \bigcup_{k=1}^{\infty} E_k,$$

where each $\Gamma_j$ is a quasicircle and each $E_k$ is a totally disconnected compact set.

Then $E$ is conformally removable if and only if every $E_k$ is conformally removable.

Still open whether the union of two conformally removable sets is conformally removable (Jones–Smirnov, 2000).
Second Main Theorem
Second Main Theorem

It is not difficult to prove that
Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.
Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.

Theorem (Y. (2015))

A circle domain $\Omega$ is conformally rigid if and only if it is quasiconformally rigid.
Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.

**Theorem (Y. (2015))**

A circle domain $\Omega$ is conformally rigid if and only if it is quasiconformally rigid.

**Corollary**
Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.

**Theorem (Y. (2015))**

A circle domain $\Omega$ is conformally rigid if and only if it is quasiconformally rigid.

**Corollary**

Let $\Omega$ be a circle domain and let $f$ be a quasiconformal mapping of the sphere which maps $\Omega$ onto another circle domain $f(\Omega)$. 
Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.

**Theorem (Y. (2015))**

A circle domain $\Omega$ is conformally rigid if and only if it is quasiconformally rigid.

**Corollary**

Let $\Omega$ be a circle domain and let $f$ be a quasiconformal mapping of the sphere which maps $\Omega$ onto another circle domain $f(\Omega)$. If $\Omega$ is conformally rigid, then $f(\Omega)$ is also conformally rigid.
An important lemma
An important lemma
An important lemma

Lemma

If $\Omega$ is quasiconformally rigid, then $\partial \Omega$ has zero area.
If $\Omega$ is quasiconformally rigid, then $\partial \Omega$ has zero area.

Assume $\partial \Omega$ has positive area.
An important lemma

Lemma

If $\Omega$ is quasiconformally rigid, then $\partial \Omega$ has zero area.

Assume $\partial \Omega$ has positive area.

- Construct a measurable function $\mu : \hat{\mathbb{C}} \to \mathbb{D}$ such that
An important lemma

Lemma

If $\Omega$ is quasiconformally rigid, then $\partial \Omega$ has zero area.

Assume $\partial \Omega$ has positive area.

- Construct a measurable function $\mu : \hat{C} \to \mathbb{D}$ such that
  - $\mu = 0$ on $\Omega$, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
Lemma

If $\Omega$ is quasiconformally rigid, then $\partial \Omega$ has zero area.

Assume $\partial \Omega$ has positive area.

- Construct a measurable function $\mu : \mathbb{C} \to \mathbb{D}$ such that
  - $\mu = 0$ on $\Omega$, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
  - $\mu$ satisfies the (strong) David condition:
    $$m(\{z \in \mathbb{C} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta e^{\frac{\alpha}{\epsilon}}} \quad (\epsilon > 0).$$
An important lemma

**Lemma**

*If Ω is quasiconformally rigid, then ∂Ω has zero area.*

Assume ∂Ω has positive area.

- Construct a measurable function \( \mu : \hat{\mathbb{C}} \to \mathbb{D} \) such that
  - \( \mu = 0 \) on \( \Omega \), \( \| \mu \|_{L^\infty(\Omega)} = 1 \)
  - \( \mu \) satisfies the (strong) David condition:
    \[
    m(\{ z \in \hat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon \}) \leq Me^{-\beta e^{\alpha \epsilon}} \quad (\epsilon > 0). 
    \]
  - \( \mu \) is **invariant** with respect to the Schottky group of \( \Omega \).
An important lemma

Lemma

If $\Omega$ is quasiconformally rigid, then $\partial \Omega$ has zero area.

Assume $\partial \Omega$ has positive area.

- Construct a measurable function $\mu : \hat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on $\Omega$, $\|\mu\|_{L^\infty(\Omega)} = 1$
  - $\mu$ satisfies the (strong) David condition:
    $$m(\{z \in \hat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta \epsilon^{\alpha}} \quad (\epsilon > 0).$$
  - $\mu$ is invariant with respect to the Schottky group of $\Omega$.
- Use David’s theorem to obtain a homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_z f / \partial_{\bar{z}} f = \mu$ a.e.
An important lemma

Lemma

If $\Omega$ is quasiconformally rigid, then $\partial \Omega$ has zero area.

Assume $\partial \Omega$ has positive area.

- Construct a measurable function $\mu : \hat{\mathbb{C}} \rightarrow \mathbb{D}$ such that
  - $\mu = 0$ on $\Omega$, $\|\mu\|_{L^\infty(\Omega)} = 1$
  - $\mu$ satisfies the (strong) David condition:
    $$m(\{z \in \hat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta\epsilon^\alpha} \quad (\epsilon > 0).$$
  - $\mu$ is invariant with respect to the Schottky group of $\Omega$.
- Use David's theorem to obtain a homeomorphism $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$
  belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_z f / \partial_{\bar{z}} f = \mu$ a.e.
- Deduce that
An important lemma

Lemma

If $\Omega$ is quasiconformally rigid, then $\partial \Omega$ has zero area.

Assume $\partial \Omega$ has positive area.

- Construct a measurable function $\mu : \hat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on $\Omega$, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
  - $\mu$ satisfies the (strong) David condition:
    $m(\{z \in \hat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta \epsilon^\alpha}$ ($\epsilon > 0$).
  - $\mu$ is invariant with respect to the Schottky group of $\Omega$.

- Use David’s theorem to obtain a homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_z f / \partial \bar{z} f = \mu$ a.e.

- Deduce that
  - $f$ is conformal on $\Omega$
An important lemma

**Lemma**

*If* $\Omega$ *is quasiconformally rigid, then* $\partial \Omega$ *has zero area.*

Assume $\partial \Omega$ has positive area.

- Construct a measurable function $\mu : \hat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on $\Omega$, $\|\mu\|_{L^\infty(\Omega)} = 1$
  - $\mu$ satisfies the (strong) David condition:
    $$m(\{z \in \hat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta \epsilon^{\alpha}} \quad (\epsilon > 0).$$
  - $\mu$ is invariant with respect to the Schottky group of $\Omega$.

- Use David’s theorem to obtain a homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_z f / \partial \bar{z} f = \mu$ a.e.

- Deduce that
  - $f$ is conformal on $\Omega$
  - $f$ is not the restriction of a quasiconformal mapping of the whole sphere
An important lemma

Lemma

If $\Omega$ is quasiconformally rigid, then $\partial \Omega$ has zero area.

Assume $\partial \Omega$ has positive area.

- Construct a measurable function $\mu : \hat{C} \to D$ such that
  - $\mu = 0$ on $\Omega$, $\|\mu\|_{L^\infty(\Omega)} = 1$
  - $\mu$ satisfies the (strong) David condition:
    $m(\{z \in \hat{C} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta e^{\alpha}}$ ($\epsilon > 0$).
  - $\mu$ is invariant with respect to the Schottky group of $\Omega$.
- Use David’s theorem to obtain a homeomorphism $f : \hat{C} \to \hat{C}$ belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_z f / \partial \bar{z} f = \mu$ a.e.
- Deduce that
  - $f$ is conformal on $\Omega$
  - $f$ is not the restriction of a quasiconformal mapping of the whole sphere
  - $f(\Omega)$ is a circle domain.
Quasiconformally rigid implies conformally rigid
Assume that $\Omega$ is quasiconformally rigid, and let $f$ be a conformal map of $\Omega$ onto another circle domain.
Assume that $\Omega$ is quasiconformally rigid, and let $f$ be a conformal map of $\Omega$ onto another circle domain.

Then $f$ is the restriction of a quasiconformal mapping $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. 
Assume that $\Omega$ is quasiconformally rigid, and let $f$ be a conformal map of $\Omega$ onto another circle domain.

- Then $f$ is the restriction of a quasiconformal mapping $g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$.
- Use the fact that $\partial \Omega$ has zero area and results of Sullivan on Kleinian groups to deduce that $g$ is a Möbius transformation.
THANK YOU!