The Analyst’s Traveling Salesman theorem in graph inverse limits

Guy C. David, NYU
Joint work with Raanan Schul, Stony Brook

AMS Spring Eastern Sectional Meeting, Stony Brook University
Special Session on Geometric Measure Theory and Its Applications

March 20, 2016
Let $E$ be a set in $\mathbb{R}^2$, and let $Q$ be a cube in $\mathbb{R}^2$. 

**\(\beta\)-numbers in $\mathbb{R}^2$**
Let $E$ be a set in $\mathbb{R}^2$, and let $Q$ be a cube in $\mathbb{R}^2$.

Define

$$\beta_E(Q) = \frac{1}{\text{diam}(Q)} \inf_{L} \sup_{x \in E \cap Q} \text{dist}(x, L)$$

where the infimum is taken over all lines $L$ in the plane.
The Analyst’s Traveling Salesman Theorem

Theorem (Jones ’90)

(a) (Upper Bound) If $\Gamma \subset \mathbb{R}^2$ is connected, then

$$\sum_{\text{dyadic cubes } Q} \beta_\Gamma (3Q)^2 \text{diam}(Q) \lesssim \mathcal{H}^1(\Gamma).$$
The Analyst’s Traveling Salesman Theorem

Theorem (Jones ’90)

(a) (Upper Bound) If \( \Gamma \subset \mathbb{R}^2 \) is connected, then

\[
\sum_{\text{dyadic cubes } Q} \beta_\Gamma(3Q)^2 \text{diam}(Q) \lesssim \mathcal{H}^1(\Gamma).
\]

(b) (Construction) If \( E \subset \mathbb{R}^2 \) is any set, then \( E \) is contained in a connected set \( \Gamma \) satisfying

\[
\mathcal{H}^1(\Gamma) \lesssim \text{diam}(E) + \sum_{\text{dyadic cubes } Q} \beta_E(3Q)^2 \text{diam}(Q).
\]
Generalizations

- Okikiolu '92: True in all $\mathbb{R}^n$. 
Generalizations

- Okikiolu '92: True in all $\mathbb{R}^n$.
- Schul '07: True in Hilbert space.
Generalizations

- Okikiolu ’92: True in all $\mathbb{R}^n$.
- Schul ’07: True in Hilbert space.
- Hahlomaa, Schul: Generalizations via Menger curvature and other quantities to metric spaces.
Generalizations

- Okikiolu ’92: True in all $\mathbb{R}^n$.
- Schul ’07: True in Hilbert space.
- Hahlomaa, Schul: Generalizations via Menger curvature and other quantities to metric spaces.
- Ferrari-Franchi-Pajot ’07: Generalization to the Heisenberg group, with $\beta$-numbers measured with respect to horizontal lines.
The Traveling Salesman Theorem in $\mathbb{R}^2$

Generalizations

- Okikiolu '92: True in all $\mathbb{R}^n$.
- Schul '07: True in Hilbert space.
- Hahlomaa, Schul: Generalizations via Menger curvature and other quantities to metric spaces.
- Ferrari-Franchi-Pajot '07: Generalization to the Heisenberg group, with $\beta$-numbers measured with respect to horizontal lines.
- Li-Schul '14, '15: Improved generalization to Heisenberg group, showing that the relevant exponent is 4, not 2.
Lessons from the Heisenberg group story

Suppose you have a metric space and you want a “geometric” traveling salesman theorem: Subsets of rectifiable curves are characterized by being quantitatively close to “lines” at most locations and scales.
Lessons from the Heisenberg group story

Suppose you have a metric space and you want a “geometric” traveling salesman theorem: Subsets of rectifiable curves are characterized by being quantitatively close to “lines” at most locations and scales. You then need:
Suppose you have a metric space and you want a “geometric” traveling salesman theorem: Subsets of rectifiable curves are characterized by being quantitatively close to “lines” at most locations and scales. You then need:

- a correct notion of “lines”, which might not be “all geodesics”, and
Lessons from the Heisenberg group story

Suppose you have a metric space and you want a “geometric” traveling salesman theorem: Subsets of rectifiable curves are characterized by being quantitatively close to “lines” at most locations and scales. You then need:

- a correct notion of “lines”, which might not be “all geodesics”, and
- the correct exponent(s).
Definition of the spaces (Cheeger-Kleiner)

Our space $X$ will be an inverse limit of connected simplicial metric graphs:

$$X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} \ldots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \ldots$$
Definition of the spaces (Cheeger-Kleiner)

Our space $X$ will be an inverse limit of connected simplicial metric graphs:

$$X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} \ldots \xleftarrow{\pi_i} X_i \xleftarrow{\pi_i} \ldots$$

For some constants $\eta > 0$, $2 \leq m \in \mathbb{N}$, $\Delta > 0$, we require that our graphs satisfy four axioms for each $i \in \mathbb{Z}$:
Definition of the spaces (Cheeger-Kleiner)
Our space $X$ will be an inverse limit of connected simplicial metric graphs:

$$X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} \ldots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \ldots$$

For some constants $\eta > 0$, $2 \leq m \in \mathbb{N}$, $\Delta > 0$, we require that our graphs satisfy four axioms for each $i \in \mathbb{Z}$:

1. $(X_0, d_0)$ is isometric to $[0, 1]$. 
Definition of the spaces (Cheeger-Kleiner)

Our space $X$ will be an inverse limit of connected simplicial metric graphs:

$$X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} \ldots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \ldots$$

For some constants $\eta > 0$, $2 \leq m \in \mathbb{N}$, $\Delta > 0$, we require that our graphs satisfy four axioms for each $i \in \mathbb{Z}$:

1. $(X_0, d_0)$ is isometric to $[0, 1]$.
2. $(X_i, d_i)$ is a nonempty connected graph with all vertices of valence at most $\Delta$, and such that every edge of $X_i$ is isometric to an interval of length $m^{-i}$ with respect to the path metric $d_i$. 
Definition of the spaces (Cheeger-Kleiner)

Our space $X$ will be an inverse limit of connected simplicial metric graphs:

$$X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} \ldots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \ldots$$

For some constants $\eta > 0$, $2 \leq m \in \mathbb{N}$, $\Delta > 0$, we require that our graphs satisfy four axioms for each $i \in \mathbb{Z}$:

1. $(X_0, d_0)$ is isometric to $[0, 1]$.
2. $(X_i, d_i)$ is a nonempty connected graph with all vertices of valence at most $\Delta$, and such that every edge of $X_i$ is isometric to an interval of length $m^{-i}$ with respect to the path metric $d_i$.
3. If $X'_i$ denotes the graph obtained by subdividing each edge of $X_i$ into $m$ edges of length $m^{-(i+1)}$, then $\pi_i$ induces a map $\pi_i : (X_{i+1}, d_{i+1}) \to (X'_i, d_i)$ which is open, simplicial, and an isometry on every edge.
Definition of the spaces (Cheeger-Kleiner)

Our space $X$ will be an inverse limit of connected simplicial metric graphs:

$$X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \cdots$$

For some constants $\eta > 0$, $2 \leq m \in \mathbb{N}$, $\Delta > 0$, we require that our graphs satisfy four axioms for each $i \in \mathbb{Z}$:

1. $(X_0, d_0)$ is isometric to $[0, 1]$.
2. $(X_i, d_i)$ is a nonempty connected graph with all vertices of valence at most $\Delta$, and such that every edge of $X_i$ is isometric to an interval of length $m^{-i}$ with respect to the path metric $d_i$.
3. If $X'_i$ denotes the graph obtained by subdividing each edge of $X_i$ into $m$ edges of length $m^{-(i+1)}$, then $\pi_i$ induces a map $\pi_i : (X_{i+1}, d_{i+1}) \to (X'_i, d_i)$ which is open, simplicial, and an isometry on every edge.
4. For every $x_i \in X'_i$, the inverse image $\pi_{i-1}^{-1}(x_i) \subset X_{i+1}$ has $d_{i+1}$-diameter at most $\eta m^{-i}$. 
Example 1: The Lang-Plaut example
Example 1: The Lang-Plaut example
Example 1: The Lang-Plaut example
Example 2: The dyadic Laakso example
Example 2: The dyadic Laakso example

![Diagram of the dyadic Laakso example]
Example 2: The dyadic Laakso example
Monotone geodesics

Analogous to horizontal lines in the Heisenberg group, these spaces have a naturally distinguished class of geodesics.

Definition

A geodesic $\gamma$ in $X$ is called monotone if $\pi_0|\gamma: \gamma \to X_0 \sim = [0,1]$ is an isometry.

G.C. David, R. Schul (2016)
Monotone geodesics

Analogous to horizontal lines in the Heisenberg group, these spaces have a naturally distinguished class of geodesics.

**Definition**

A geodesic $\gamma$ in $X$ is called **monotone** if $\pi_0|_{\gamma} : \gamma \to X_0 \cong [0, 1]$ is an isometry.
Monotone geodesics

Analogous to horizontal lines in the Heisenberg group, these spaces have a naturally distinguished class of geodesics.

**Definition**

A geodesic $\gamma$ in $X$ is called **monotone** if $\pi_0|_{\gamma} : \gamma \to X_0 \cong [0, 1]$ is an isometry.
Let $E$ be a subset of $X$, and let $B$ be a ball in $X$.

**Definition**

We define

$$\beta_E(B) = \frac{1}{\text{diam}(B)} \inf_L \sup_{x \in E \cap B} \text{dist}(x, L)$$

where the supremum is taken over all **monotone geodesics** $L$ in $X$. 
The main results

The upper bound

Fix a space $X$ as above, and an appropriate $m$-adic system $G$ of balls in $X$.

**Theorem (Upper bound)**

For every $p > 1$, there is a constant $C_p$ such that, if $\Gamma \subset X$ is connected, then

$$\sum_{B \in G} \beta_\Gamma(B)^p \text{diam}B \leq C_p \mathcal{H}^1(\Gamma).$$

The constant $C_p$ depends only on $p$ and the constants associated to the construction of $X$. 
The main results

The upper bound

Fix a space $X$ as above, and an appropriate $m$-adic system $G$ of balls in $X$.

**Theorem (Upper bound)**

For every $p > 1$, there is a constant $C_p$ such that, if $\Gamma \subset X$ is connected, then

$$\sum_{B \in G} \beta_\Gamma(B)^p \text{diam} B \leq C_p \mathcal{H}^1(\Gamma).$$

The constant $C_p$ depends only on $p$ and the constants associated to the construction of $X$.

The exponent is sharp: there is a counterexample for $p = 1$. 
The construction

Fix a space $X$ as above, and an appropriate $m$-adic system $G$ of balls in $X$.

Theorem (Construction)

*There are constants $C > 1$ and $\epsilon > 0$, depending only on the data of $X$, with the following property: Let $E \subset X$ be compact. Then there is a compact connected set $\Gamma \subset X$ containing $E$ such that*

$$
\mathcal{H}^1(\Gamma) \leq C \left( \text{diam}(E) + \sum_{B \in G, \beta_E(B) \geq \epsilon} \text{diam}(B) \right).
$$
The main results

The construction

Fix a space $X$ as above, and an appropriate $m$-adic system $G$ of balls in $X$.

Theorem (Construction)

*There are constants $C > 1$ and $\epsilon > 0$, depending only on the data of $X$, with the following property: Let $E \subset X$ be compact. Then there is a compact connected set $\Gamma \subset X$ containing $E$ such that*

\[
\mathcal{H}^1(\Gamma) \leq C \left( \text{diam}(E) + \sum_{B \in G, \beta_E(B) \geq \epsilon} \text{diam}(B) \right).
\]

Remark: This implies that

\[
\mathcal{H}^1(\Gamma) \leq C_p \left( \text{diam}(E) + \sum_{B \in G} \beta_E(B)^p \text{diam}(B) \right),
\]

where $C_p$ depends only on $p > 0$ and the data of $X$. 

---

Idea of the proof in the upper bound

- Modifying ideas of Okikiolu, we divide the collection of $m$-adic balls $G$ into two types:
Idea of the proof in the upper bound

- Modifying ideas of Okikiolu, we divide the collection of $m$-adic balls $\mathcal{G}$ into two types:
- $\mathcal{G}_1 \subset \mathcal{G}$ consists of those balls in which the size of the $\beta$-number is due to the presence of multiple individually flat but far pieces of the curve.
Idea of the proof in the upper bound

- Modifying ideas of Okikiolu, we divide the collection of $m$-adic balls $\mathcal{G}$ into two types:
  - $\mathcal{G}_1 \subset \mathcal{G}$ consists of those balls in which the size of the $\beta$-number is due to the presence of multiple individually flat but far pieces of the curve.
  - $\mathcal{G}_2 \subset \mathcal{G}$ consists of those balls in which the size of the $\beta$-number is due to one large wiggly piece of the curve.
Idea of the proof in the upper bound

- Modifying ideas of Okikiolu, we divide the collection of $m$-adic balls $\mathcal{G}$ into two types:
- $\mathcal{G}_1 \subset \mathcal{G}$ consists of those balls in which the size of the $\beta$-number is due to the presence of multiple individually flat but far pieces of the curve.
- $\mathcal{G}_2 \subset \mathcal{G}$ consists of those balls in which the size of the $\beta$-number is due to one large wiggly piece of the curve.

(a) A $\mathcal{G}_1$ ball.

(b) A $\mathcal{G}_2$ ball.
Idea of the proof in the upper bound

- The sum over the flat balls $G_1$ can be controlled by a very general martingale construction of Schul.
Idea of the proof in the upper bound

- The sum over the flat balls $G_1$ can be controlled by a very general martingale construction of Schul.
- The sum over the non-flat balls $G_2$ is controlled by parametrizing the curve $\Gamma$ by $\gamma : [0, 1] \to X$ and using a quantitative differentiation result for the function $\pi_0 \circ \gamma : [0, 1] \to [0, 1]$. 

G.C. David, R. Schul (2016) 
Analyst’s TSP in graph limits 
March 20, 2016
The sum over the flat balls $G_1$ can be controlled by a very general martingale construction of Schul.

The sum over the non-flat balls $G_2$ is controlled by parametrizing the curve $\Gamma$ by $\gamma : [0, 1] \to X$ and using a quantitative differentiation result for the function $\pi_0 \circ \gamma : [0, 1] \to [0, 1]$.

The idea here is that, due to the discrete approximation of the space, if the parametrization $\gamma$ passes through a “non-flat ball”, $\pi_0 \circ \gamma$ must backtrack, and there is a quantitative bound on how much a real-valued Lipschitz function can backtrack.
Idea in the proof of the construction

Fix a set $E \subset X$.

- One tries to inductively build connected constructions

$$\Gamma_i \subset X_i$$

that are progressively nearer to $\pi_i(E)$. 
Idea in the proof of the construction

Fix a set $E \subset X$.

- One tries to inductively build connected constructions
  
  $$\Gamma_i \subset X_i$$
  
  that are progressively nearer to $\pi_i(E)$.

- The idea is to use $\beta$-numbers to inductively “lift” the previous construction $\Gamma_{i-1} \subset X_{i-1}$ to a connected construction $\Gamma_i \subset X_i$ without adding too much length.
Idea in the proof of the construction

Fix a set $E \subset X$.

- One tries to inductively build connected constructions
  $$\Gamma_i \subset X_i$$
  that are progressively nearer to $\pi_i(E)$.

- The idea is to use $\beta$-numbers to inductively “lift” the previous construction $\Gamma_{i-1} \subset X_{i-1}$ to a connected construction $\Gamma_i \subset X_i$ without adding too much length.

- At locations in $\Gamma_{i-1}$ with large $\beta$, it is clear what to do: just take all possible lifts.
Idea in the proof of the construction

Fix a set $E \subset X$.

- One tries to inductively build connected constructions
  $$\Gamma_i \subset X_i$$
  that are progressively nearer to $\pi_i(E)$.

- The idea is to use $\beta$-numbers to inductively “lift” the previous construction $\Gamma_{i-1} \subset X_{i-1}$ to a connected construction $\Gamma_i \subset X_i$ without adding too much length.

- At locations in $\Gamma_{i-1}$ with large $\beta$, it is clear what to do: just take all possible lifts.

- At locations with small $\beta$, one must be careful to take an essentially optimal lift and maintain its connectedness to the rest of the curve. This is where the majority of the technical problems come in.