The geometry of RNP Lipschitz differentiability spaces, I

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19th March 2016

Lipschitz differentiability spaces

We will consider generalisations of Rademacher's theorem to metric measure spaces (X, d, μ). Fix a Lipschitz $\varphi \colon X \to \mathbb{R}^n$.

Definition

A function $f \colon X \to \mathbb{R}$ is differentiable at x_0 if there exists a unique, linear $Df(x_0) \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) - f(x_0) = Df(x_0)(\varphi(x) - \varphi(x_0)) + o(d(x, x_0)).$$

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Definition

A metric measure space is an n-dimensional *RNP Lipschitz differentiability space* if every Lipschitz $f\colon X \to V$ is differentiable μ a.e. for every V with the Radon Nikodym property.

V has the RNP if every Lipschitz $\gamma\colon [0,1] \to V$ is differentiable (Lebesgue) a.e.

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Examples: Heisenberg group, Laakso space.

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A probability measure $\mathbb{P}$ on Γ and measures $\mu_\gamma \ll \mathcal{H}^1 \llcorner \gamma$ form an *Alberti representation* of μ if

$$\mu(B) = \int \mu_\gamma(B) \mathrm{d}\mathbb{P}(\gamma)$$

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Gives a *partial derivative* $(f \circ \gamma)'(t)$ of any Lipschitz function f at μ-a.e. x.

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Theorem (B '12)

(X, d, μ) *is an n-dimensional Lipschitz differentiability space if and only if* μ *has a universal collection of n Alberti representatations.*

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- $b - a \leq Cd(x, x_0)$ and $|[a,b] \setminus \gamma^{-1}(S)| \leq \epsilon d(x, x_0)$.

We say that such Alberti representations *connect points* in X.

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Now let $f\colon X \to \mathbb{R}$ be Lipschitz and for $D \in \mathbb{R}^n$ and $\epsilon > 0$ set

$$S = S_{D,\epsilon} = \{y \in X : \|\nabla f(y) - D \cdot \nabla \varphi(y)\| < \epsilon\}.$$

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(For any $\epsilon > 0$, we can cover almost all of X by sets of this form.) Then by using the Fundamental theorem of calculus along the curve γ,

$$\begin{aligned} f(x) - f(x_0) &= \int_{\gamma^{-1}(S)} (f \circ \gamma)' + \epsilon d(x, x_0) \operatorname{Lip} f O(1) \\ &= \int_{\gamma^{-1}(S)} D \cdot (\varphi \circ \gamma)' + \epsilon d(x, x_0)(C + \operatorname{Lip}) O(1) \\ &= D \cdot (\varphi(x) - \varphi(x_0)) + \epsilon d(x, x_0)(\operatorname{Lip} \varphi + C + \operatorname{Lip} f) O(1) \end{aligned}$$

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Let $\epsilon \to 0$, $D \to Df(x_0)$ for μ-a.e x_0. Also works for RNP valued f.

A characterisation of RNP-LDS

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- We need RNP targets, specifically an ℓ_1 sum of ℓ_p spaces ($p \to \infty$).
- It is an open question whether LDS $\Leftrightarrow$ RNP-LDS.

Idea of the proof

- Negating "Alberti representations connect points" (and some measure theory) gives a $\beta > 0$ such that, for every $\epsilon > 0$,

$$\rho_\epsilon(x, x_0) := \inf_\gamma \epsilon|\operatorname{dom}\gamma| + |[a, b] \setminus \operatorname{dom}\gamma|$$

(γ a concatenation of curves in Γ_i joining x to x_0) satisfies

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- They cannot be glued together to form a single function that is bad on a set of large measure.
- Instead, each function goes into a component of an ℓ_p valued function. Combining over $\epsilon \to 0$ requires a further ℓ_1 sum.