Singular sets of n-uniform measures

A. Dali Nimer (University of Washington)

AMS Spring Eastern Sectional Meeting, State University of New York at Stony Brook, NY

March 19th, 2016
Besicovitch (1938) Let $E \subset \mathbb{R}^2$, $0 < \mathcal{H}^1(E) < \infty$ and for $\mathcal{H}^1$ almost every $x \in E$,

$$\lim_{r \to 0} \frac{\mathcal{H}^1(E \cap B(x, r))}{2r} = 1.$$ 

Then

$E$ is $1 -$ rectifiable.
Theorem (Preiss)

Let $\Phi$ be a Radon measure on $\mathbb{R}^d$. Then $\Phi$ is $n$-rectifiable (i.e. $\Phi \ll \mathcal{H}^n$ and that $\Phi(\mathbb{R}^d \setminus E) = 0$ for some $n$-rectifiable set $E$) if and only if for $\Phi$ almost every $x$, $\Theta^n(\Phi, x) = \lim_{r \to 0} \frac{\Phi(B(x,r))}{\omega_n r^n}$ exists and

$$0 < \Theta^n(\Phi, x) < \infty.$$
Definition

Let $\Phi$ be a Radon measure on $\mathbb{R}^d$, $x$ a point in its support such that $\Theta^n(\Phi, x)$ is positive and finite. We say that $\lambda$ is a tangent measure of $\Phi$ at $x$ and denote $\lambda \in Tan(\Phi, x)$ if $\lambda \neq 0$ and there exists a sequence of positive radii $(r_i)$, with $r_i \downarrow 0$ such that:

$$
\Phi_{x, r_i} \rightharpoonup \lambda \text{ as } i \to \infty,
$$

where the convergence is the weak convergence of measures and

$$
\Phi_{x, r} = r^{-n} T_{x, r} \Phi
$$

is the push-forward of $\Phi$ by the homothecy $T_{x, r}(y) = \frac{y-x}{r}$.
Definition

Let $\mu$ be a Radon measure in $\mathbb{R}^d$. We say $\mu$ is $n$-uniform if there exists $c > 0$ such that for all $x \in spt(\mu)$, $r > 0$:

$$\mu(B(x, r)) = cr^n.$$
(**Kirchheim-Preiss**) The support of an \( n \)-uniform measure is an analytic variety. Remark: By Lojasiewicz’ structure theorem, its singular set has Hausdorff dimension at most \((n - 1)\).

- (Preiss) \( n = 1, \, d \geq 1: \mathcal{H}^1_\mathbb{R} \).
- (Preiss) \( n = 2, \, d \geq 2: \mathcal{H}^2_\mathbb{R}^2 \).
(Kirchheim-Preiss) The support of an $n$-uniform measure is an analytic variety. Remark: By Lojasiewicz’ structure theorem, its singular set has Hausdorff dimension at most $(n - 1)$.

(Preiss) $n = 1, d \geq 1$: $\mathcal{H}^1 \cap \mathbb{R}$.

(Preiss) $n = 2, d \geq 2$: $\mathcal{H}^2 \cap \mathbb{R}^2$. 
(Kirchheim-Preiss) The support of an $n$-uniform measure is an analytic variety. Remark: By Lojasiewicz’ structure theorem, its singular set has Hausdorff dimension at most $(n - 1)$.

(Preiss) $n = 1, d \geq 1$: $\mathcal{H}^1 \subseteq \mathbb{R}$.

(Preiss) $n = 2, d \geq 2$: $\mathcal{H}^2 \subseteq \mathbb{R}^2$. 
(Kowalski-Preiss) In $\mathbb{R}^4$, consider the cone

$$ C = \left\{ (x_1, x_2, x_3, x_4); x_4^2 = x_1^2 + x_2^2 + x_3^2 \right\}. $$

Then the measure $\mathcal{H}^3 \llcorner C$ is 3-uniform.

(Kowalski-Preiss) $d = n + 1$ the support of an $n$-uniform measure in $\mathbb{R}^{n+1}$ can only be an $n$-plane or (up to rotation) $\mathbb{R}^{n-3} \times C$
(Kowalski-Preiss) In $\mathbb{R}^4$, consider the cone

$$C = \left\{ (x_1, x_2, x_3, x_4); x_4^2 = x_1^2 + x_2^2 + x_3^2 \right\}.$$ 

Then the measure $\mathcal{H}^3 \ll C$ is 3-uniform.

(Kowalski-Preiss) $d = n + 1$ the support of an $n$-uniform measure in $\mathbb{R}^{n+1}$ can only be an $n$-plane or (up to rotation) $\mathbb{R}^{n-3} \times C$. 

A. Dali Nimer (University of Washington) 

Singular sets of n-uniform measures
Questions:

- How large is the singular set of \( n \)-uniform measures?
- Are there other \( n \)-uniform measures?
- Can we classify them?
Questions:

- How large is the singular set of \(n\)-uniform measures?
- Are there other \(n\)-uniform measures?
- Can we classify them?
Questions:

- How large is the singular set of $n$-uniform measures?
- Are there other $n$-uniform measures?
- Can we classify them?
Questions:

- How large is the singular set of $n$-uniform measures?
- Are there other $n$-uniform measures?
- Can we classify them?
**Definition**

Let $\mu$ be an $n$-uniform measure in $\mathbb{R}^d$.

- We call $x_0$ a flat point of $\mu$ if there exists a unique $n$-plane $V_{x_0}$ such that $\text{Tan}(\mu, x_0) = \{c\mathcal{H}^n \cap V_{x_0} ; c > 0\}$.
- We denote by $S_\mu$ the singular set of $\mu$ defined as:

$$S_\mu = \{x \in spt(\mu) ; x \text{ is not a flat point} \} .$$

- We say $\mu$ is a conical $n$-uniform measure if for $A \subset \mathbb{R}^d$, $r > 0$,

$$\mu(rA) = r^n \mu(A) .$$
Theorem (N. 2015)

Let $\mu$ be an $n$-uniform measure in $\mathbb{R}^d$, $n \geq 3$ and $S_\mu$ be its set of singularities. Then:

$$\dim_H(S_\mu) \leq n - 3.$$
This bound is sharp. Indeed, taking $\mu = \mathcal{H}^n_L(\mathbb{R}^{n-3} \times C)$ we have

$$\dim_{\mathcal{H}}(S_\mu) = n - 3.$$
Theorem (N. 2015)

Let $\nu$ be a conical 3-uniform measure, $\Omega = \text{spt}(\nu) \cap \mathbb{S}^{d-1}$, and $\sigma = \mathcal{H}^2 \lfloor \Omega$. Then for all $x \in \Omega$, for $0 \leq r \leq 2$:

$$\sigma(B(x, r)) = \pi r^2.$$
Theorem (N. 2015)

Let $\mu$ be an $n$-uniform measure in $\mathbb{R}^d$, $x_0 \in \text{spt}(\mu)$. Let $\nu$ be a tangent to $\mu$ at $x_0$ and let $r_j$ be a sequence of positive radii so that

$$\mu_{x_0, r_j} \rightharpoonup \nu.$$ 

Then if $\{x_j\} \subset S_\mu$, there exists $y \in S_\nu$ so that:

$$\frac{x_j - x_0}{r_j} \rightarrow y.$$
Questions:

- How large is the singular set of $n$-uniform measures?
- Are there other $n$-uniform measures?
Questions:

- How large is the singular set of $n$-uniform measures?
- Are there other $n$-uniform measures?
Questions:
- How large is the singular set of $n$-uniform measures?
- Are there other $n$-uniform measures?
Projects

- Obtain full description of 3-uniform conical measures.
- Are there other examples of 3-uniform measures?
Projects

- Obtain full description of 3-uniform conical measures.
- Are there other examples of 3-uniform measures?
Thank you!