Singular sets of n-uniform measures

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Besicovitch(1938) Let $E \subset \mathbb{R}^2, 0 < \mathcal{H}^1(E) < \infty$ and for $\mathcal{H}^1$ almost every $x \in E$,

$$\lim_{r \to 0} \frac{\mathcal{H}^1(E \cap B(x,r))}{2r} = 1.$$

Then

$$E \text{ is } 1-\text{rectifiable.}$$

Theorem (Preiss)

Let Φ be a Radon measure on $\mathbb{R}^d$. Then Φ is n-rectifiable (i.e. $\Phi << \mathcal{H}^n$ and that $\Phi(\mathbb{R}^d \backslash E) = 0$ for some n-rectifiable set E) if and only if for Φ almost every x, $\Theta^n(\Phi, x) = \lim_{r\to 0} \frac{\Phi(B(x,r))}{\omega_n r^n}$ exists and

$$0 < \Theta^n(\Phi, x) < \infty.$$

Definition

Let Φ be a Radon measure on $\mathbb{R}^d$, x a point in its support such that $\Theta^n(\Phi, x)$ is positive and finite. We say that λ is a tangent measure of Φ at x and denote $\lambda \in Tan(\Phi, x)$ if $\lambda \neq 0$ and there exists a sequence of positive radii (r_i), with $r_i \downarrow 0$ such that:

$$\Phi_{x,r_i} \rightharpoonup \lambda \text{ as } i \to \infty,$$

where the convergence is the weak convergence of measures and

$$\Phi_{x,r} = r^{-n} T_{x,r} \Phi$$

is the push-forward of Φ by the homothecy $T_{x,r}(y) = \frac{y-x}{r}$.

Definition

Let μ be a Radon measure in $\mathbb{R}^d$. We say μ is n-uniform if there exists $c > 0$ such that for all $x \in spt(\mu)$, $r > 0$:

$$\mu(B(x, r)) = cr^n.$$

- (**Kirchheim-Preiss**)The support of an n-uniform measure is an analytic variety. Remark: By Lojasiewicz' structure theorem, its singular set has Hausdorff dimension at most $(n-1)$.
- (**Preiss**) $n = 1$, $d \geq 1$: $\mathcal{H}^1 \llcorner \mathbb{R}$.
- (**Preiss**) $n = 2$, $d \geq 2$: $\mathcal{H}^2 \llcorner \mathbb{R}^2$.

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- (**Kowalski-Preiss**) In $\mathbb{R}^4$, consider the cone

$$C = \left\{(x_1, x_2, x_3, x_4); x_4^2 = x_1^2 + x_2^2 + x_3^2\right\}.$$

Then the measure $\mathcal{H}^3 \llcorner C$ is 3-uniform.

- (**Kowalski-Preiss**) $d = n + 1$ the support of an n-uniform measure in $\mathbb{R}^{n+1}$ can only be an n-plane or (up to rotation) $\mathbb{R}^{n-3} \times C$

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- How large is the singular set of n-uniform measures?
- Are there other n-uniform measures?
- Can we classify them?

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Definition

Let μ be an n-uniform measure in $\mathbb{R}^d$.

- We call x_0 a flat point of μ if there exists a unique n-plane V_{x_0} such that $Tan(\mu, x_0) = \{c\mathcal{H}^n \llcorner V_{x_0}; c > 0\}$.
- We denote by $\mathcal{S}_\mu$ the singular set of μ defined as:

$$\mathcal{S}_\mu = \{x \in spt(\mu); x \text{ is not a flat point }\}.$$

- We say μ is a conical n-uniform measure if for $A \subset \mathbb{R}^d$, $r > 0$,

$$\mu(rA) = r^n \mu(A).$$

Theorem (N. 2015)

Let μ be an n-uniform measure in $\mathbb{R}^d$, $n \geq 3$ and $\mathcal{S}_\mu$ be its set of singularities. Then:

$$dim_{\mathcal{H}}(\mathcal{S}_\mu) \leq n - 3.$$

This bound is sharp. Indeed, taking $\mu = \mathcal{H}^n \llcorner (\mathbb{R}^{n-3} \times C)$ we have

$$dim_{\mathcal{H}}(\mathcal{S}_\mu) = n - 3.$$

Theorem (N. 2015)

Let ν be a conical 3-uniform measure, $\Omega = spt(\nu) \cap \mathbb{S}^{d-1}$, and $\sigma = \mathcal{H}^2 \llcorner \Omega$. Then for all $x \in \Omega$, for $0 \leq r \leq 2$:

$$\sigma(B(x,r)) = \pi r^2.$$

Theorem (N. 2015)

Let μ be an n-uniform measure in $\mathbb{R}^d$, $x_0 \in spt(\mu)$. Let ν be a tangent to μ at x_0 and let r_j be a sequence of positive radii so that

$$\mu_{x_0,r_j} \rightharpoonup \nu.$$

Then if $\{x_j\} \subset \mathcal{S}_\mu$, there exists $y \in \mathcal{S}_\nu$ so that:

$$\frac{x_j - x_0}{r_j} \to y.$$

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Projects

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- Are there other examples of 3-uniform measures?

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Thank you!